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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

The Language of Mathematics

"Under the auspices of the Galois Institute of Mathematics of Long Island University," on May 6, 1939, was broadcast a unique dialogue between Cassius J. Keyser and Sarah V. Keyser. This dialogue, reproduced in the June issue of Scripta Mathematica under the title *The Role of Mathematics in the Tragedy of Modern Culture,* in a broad way cannot fail to be definitely stimulating and suggestive to those whose dreams are of steadily widening circles of mathematical culture and knowledge among all people.

On the other hand, certain assumptions in the dialogue must be shocking to those who dream of an era when mathematics shall be a universal value, not merely as culture but—what is of vastly greater importance—as a *method,* as a *discipline,* as a *technique.* And this sense of shock is soothed but little by the evident concern of Professor Keyser that ways and means be initiated and promoted for mitigating the "Tragedy of Modern Culture." This "tragedy" he describes as the steadily widening gulf between the great army of non-mathematical intellectuals and those who, being mathematicians, speak or write mathematics in the *language* of *mathematics.* Professor Keyser would mitigate the tragedy through popular expositions of the mathematical sciences by those gifted with both the ability and the desire to expound in terms of everyday language—the language of the man on the street. Now the basic defining principles of mathematics must ever be those of *some logic.* Whether that be two-valued or many-valued, whether the canon of an excluded middle be refused or accepted by one or by many—mathematics is *nothing* if not a *logical consistency.* In the light of this, that situation is truly a hard one in which the benefits of the richest of all logical disciplines—those of mathematics—must, perforce, be denied to all who, while speaking an everyday language, yet speak not the language of mathematics.

The genius of the situation is obvious: Too many speech symbols of the man on the street are deficient in uniqueness and clarity of meaning. Hence a language for mathematics must be created. Yet it is tragic indeed if the blessings of logical deduction must be confined to this created language of mathematics. Nor do they need to be. This modern tragedy of culture would not only be mitigated but might suffer near-annihilation if the logistic of mathematics were applied, even in part to ordinary language. Semantics might then take its place as a branch of mathematics. Editor Seidlin's question *WHAT PRICE ISOLATION* would not then be in order.

S. T. Sanders.

The Calculation of Planetary Motions

By EDGAR W. WOOLARD
Washington, D. C.

From time immemorial, man, awed and fascinated by the beauty and grandeur of the heavens, has watched the panorama of the night sky; and down through the ages, the watchers of the skies have ceaselessly tried to interpret the impressive celestial phenomena which were observed.

In the earliest times, before astronomical observations had become extensive or accurate, many of these phenomena could easily be explained in a simple, naive way. The daily rising and setting of the sun, moon, and stars were at first naturally ascribed to an actual motion of these bodies around a stationary earth: The fixed stars—which move uniformly across the sky, night after night, always grouped into the familiar and unchanging constellation figures—came to be regarded as immovably attached to a great rotating crystal sphere, centered at the earth; the sun and the moon—which, during their daily motion across the heavens, continually move slowly eastward among the stars and complete a circuit of the constellations in, respectively, a year and a month, practically in great circle paths—either were placed on separate spheres that rotated at different rates, or else were supposed to move in free space in circular orbits around the earth.

Some of the phenomena, however, could not so readily be explained. In particular, the intricate and irregular motions of the planets among the fixed stars baffled for ages all attempts to explain them satisfactorily; one after another of a long succession of theories had to be abandoned as observations accumulated and revealed additional facts about the planetary motions that could not easily be accounted for by existing theories.*

It was not until the work of Kepler in the early seventeenth century that a satisfactory explanation was found for the observed apparent motions of the planets: If is of course a familiar fact that Copernicus had previously, in the sixteenth century, maintained that the planets, including the earth, revolve around the sun; but Copernicus thought the planetary orbits were circular, while Kepler showed

*See J. L. E. Dreyer, *History of the Planetary Systems from Thales to Kepler*. Cambridge Press, 1906.

they are ellipses. From a lengthy and laborious study of the long series of accurate observations that had been accumulated by Tycho Brahe, Kepler empirically deduced three laws which completely described the motions of the planets as then known. The First Law states that the orbit of each planet is an ellipse, with the sun at one of the foci, although in most cases the eccentricity of the ellipse is very small (that of the earth's orbit, e. g., is 0.0167); the Second Law states that the speed of a planet in its orbit varies in such a way that during any two equal intervals of time, the areas swept out by the radius vector to the sun are equal; while according to the Third Law, the squares of the periods of revolution of any two planets are in the same ratio as the cubes of their *mean distances* from the sun (i. e., the cubes of the semi-major axes of their orbits).

Near the end of the seventeenth century, Sir Isaac Newton explained Kepler's laws as necessary consequences of the Law of Gravitation—that every particle of matter in the universe attracts every other particle with a force that is directed along the line joining them, and that varies directly as the product of their masses and inversely as the square of the intervening distance—although this very explanation revealed that Kepler's laws could not be quite exact: It is easily shown that two concentrically homogeneous spherical bodies will each move, under their mutual gravitational attractions alone, in a conic section about the center of mass of the system, and that therefore each body moves *relative to the other* in a conic section also (the species of conic depends on the *initial conditions*). Thus, if the solar system were composed of only the sun and a single planet, and both were spherical and concentrically homogeneous, then the planet would move about the sun in an exactly elliptical orbit, that would never change its form, size, or position in space. Actually, however, each of the planets moves in an exceedingly irregular and ever changing path, because of the additional attractions exerted by the numerous other bodies in the solar system besides the sun; but the mass of the sun is so much greater than that of any of the other bodies that it dominates the entire system. Except for a few of the minor planets and occasionally a comet, none of the departures from regular elliptic motion is very large; and the irregularities may always be treated as comparatively minor disturbances superimposed on a motion that conforms to Kepler's laws. These disturbances of Keplerian motion are technically known as perturbations; in addition, an important correction of Kepler's Third Law is required because of the motion of the sun about the center of mass of the sun and each planet, but this is easily accomplished by merely modifying the value of the con-

stant of proportionality in the Law of Gravitation. Similarly, the planets dominate their respective satellite systems; but the attractions of other bodies, and the departures of the planets from sphericity, produce perturbations in the motions of the satellites which in some cases are quite large.

The succession of contributions by Tycho Brahe, Kepler, and Newton constitutes a classic example of Scientific Method: From Newton's Law of Gravitation and the general Laws of Motion (also formulated by Newton on the basis of his own and Galileo's work), has been developed Celestial Mechanics, one of the greatest achievements of the human intellect. On the basis of the single fact that each of the celestial bodies is attracted by all the others with forces given by the Law of Gravitation, the astronomer calculates their motions far into the future, and is able to determine with unerring precision what the aspect of the heavens will be on a distant date yet to come, or what it was at a given time in the remote past. Probably no other scientific law has been subjected to as severe and extensive a test as that provided for the Law of Gravitation by the comparison of the theoretically calculated motions of the celestial bodies with the accurate observations of modern precision astronomy; and no law has been more exactly verified. In 1909, Halley's Comet, after having been invisible for nearly 75 years during the course of its long journey out beyond the orbit of Neptune, was found on a photographic plate within less than the breadth of a pinhead of its calculated position.

The calculation of how each of the celestial bodies moves under the Newtonian gravitational attractions of all the others is a difficult and intricate problem. The longer and the more accurately the motions are observed, the more complicated the small irregularities are found to be; the mathematical theories of the celestial motions are of a complexity almost beyond conception, and the length of some of the calculations involved is appalling. Single formulæ that fill dozens of printed quarto pages, series expansions in which hundreds of terms must be used, and single calculations that require years to accomplish, are not unusual in Celestial Mechanics.

Nevertheless, it is easily possible with only an elementary knowledge of mathematics and mechanics to understand the principles by which the astronomer accomplishes his remarkable achievements, and in the case of some of the simpler problems to become able to perform the computations for oneself and even to do amateur work of real value. Not only is this a source of incomparable pleasure and satisfaction, but moreover the practical solutions of the problems of mathematical astronomy contain many valuable lessons for the student of mathematics and its applications to physics in general.

To specify the position of a celestial body in the sky, the astronomer constructs, on the apparent sphere of the heavens, circles of reference entirely analogous to the meridians and parallels which the geographer imagines on the surface of the earth: The two points that mark the ends of the axis about which the celestial sphere seems to rotate daily are known as the celestial poles, the northern one of which is approximately marked by the familiar Pole Star; the great circle midway between the poles is the celestial equator. The apparent annual path of the sun among the stars is the ecliptic, and is the intersection of the plane of the earth's orbit with the celestial sphere; the two points where the ecliptic intersects the equator are called equinoxes, of which the one where the sun crosses the equator in the spring is the vernal equinox. Great circles through the poles, corresponding to meridians on the earth, are known as hour circles, and angular distance eastward from the vernal equinox (corresponding to terrestrial east longitude) is the Right Ascension; while angular distance north or south from the equator (corresponding to terrestrial latitude) is Declination. Right Ascension and Declination are the coordinates instrumentally measured in astronomical observation; from them may be trigonometrically computed the coordinates in another system that is also extensively used, and in which the ecliptic is the fundamental circle instead of the equator: Angular distance eastward from the vernal equinox along the ecliptic is known as Celestial Longitude, while angular distance north or south from the ecliptic is Celestial Latitude.

The apparent motion of a planet among the fixed stars is the result partly of its actual motion in space and partly of the fact that the earth, from which the planet is viewed, is itself moving. The two effects are treated separately by first considering the space motion, relative to the sun, of each planet, including the earth, and afterward the apparent motion on the celestial sphere as seen from the moving earth.

Since the perturbations are in general so exceedingly small, it is an advantageous and commonly employed procedure to obtain a first close approximation to the motion in space by neglecting the perturbations and considering a given planet to be moving under only the attraction of the sun; and then subsequently to apply corrections for the effect of the attractions of the other planets. Although gravitational theory shows that, under the attraction of only the sun, a planet will move in an elliptical orbit, there is of course no way of determining apriori from theory the form, size, and position in space of the orbit of any particular planet—these must be determined from observation.

The planets Mercury, Venus, Mars, Jupiter and Saturn have been known, and their motions observed, since remote antiquity. From these centuries of observation, it is possible by simple, though laborious, geometrical methods to make an accurate map of the elliptical orbits to which their actual paths closely conform, which, however, lacks a scale of miles; this procedure was employed by Kepler. The measurement of any one distance anywhere in the system will supply the scale of the map; and can be accomplished by selecting a comparatively nearby body at a favorable time, and observing its parallax or displacement as seen from two points a known distance apart.

A very different problem is presented when a previously unknown body is discovered in the heavens. Prior to 1801, the only such bodies that ever had to be considered were comets; but on January 1, 1801 the first of the tiny minor planets that move mostly between the orbits of Mars and Jupiter was sighted. Up to the present time, several hundred appearances of comets have been accurately observed, and more than 1400 minor planets have become known, while Uranus, Neptune and Pluto have been added to the list of major planets known to the ancients. Methods have had to be devised for treating all these bodies.

By means of the gravitational theory of celestial motions, it is possible, as first shown by Newton, to determine a preliminary orbit for any object very shortly after its discovery, by using the first few observations of its position in the sky as *initial conditions* to evaluate the constants of integration in the solution of the differential equations of motion:

From the fact that a conic requires five conditions for its determination, while the location of a particular point on the conic requires one additional condition, it is evident that six independent quantities will be necessary for the complete description of a planetary orbit—five to fix the orbit itself, and one to locate the planet therein—and hence six independent data of observation are needed for this purpose. Now, the differential equations for the motion of a planet relative to the sun, in rectangular coordinates with origin at the sun, obtained by equating acceleration in a *fixed* coordinate system to gravitational force per unit mass and then transforming to coordinates with origin at the sun, are*

$$\frac{d^2x}{dt^2} = -k^2\,(1+m)\,\frac{x}{r^3}, \quad \frac{d^2y}{dt^2} = -k^2\,(1+m)\,\frac{y}{r^3},$$

$$\frac{d^2z}{dt^2} = -k^2\,(1+m)\,\frac{z}{r^3},$$

*For derivation, see F. R. Moulton, *Celestial Mechanics*, 2 ed., New York, 1914, pp. 140-144.

in which r is distance of the planet from the sun, m is the mass of the planet in terms of the sun's mass as unity, and k^2 is the constant of proportionality in the Law of Gravitation; these equations form a sixth order system, and hence their solution involves six constants of integration to be evaluated from observation.

In practice, the quantities usually used to specify an elliptic orbit are: the length of its semi-major axis, a (known as the mean distance, though it is not the average distance with respect to time); its eccentricity, e; the inclination, i, of its plane to the plane of the ecliptic; the longitude of its ascending node, ☊, i.e., the arc of the ecliptic from the vernal equinox to the point on the line of intersection of the orbital plane with the ecliptic plane (line of nodes) where the planet crosses from the south side of the ecliptic to the north; and the longitude of the perihelion or closest point to the sun, π, which is defined as the sum of the longitude of the node and the arc ω of the orbit from the ascending node to the perihelion and hence is not a geometric angle. The position of the planet in its orbit is fixed by specifying the time, T, of perihelion passage, or some equivalent quantity.

These six quantities are called the *elements* of the orbit. They may readily be expressed in terms of the six constants of integration, or in terms of any desired set of convenient functions of these constants—in particular, they may be expressed in terms of either the coordinates and their first derivatives (position and velocity) at a given instant, or of the coordinates at two given instants. Either of these latter sets of quantities may therefore be used as the *initial conditions* or so-called *intermediate elements*, to be evaluated from observations and used to determine the orbital elements. Since one observation of position gives two quantities—the two coordinates on the celestial sphere—it is evident that to get six data, at least three observations at different times are necessary; more are required under certain exceptional circumstances when some of the formulæ involved in the computations of the elements become indeterminate.

It is comparatively easy to devise methods by which the problem may theoretically be solved; but it is not such a simple matter to construct a solution that will meet the requirements of practical astronomy: A theoretical solution, no matter how elegant, of the abstract mathematical problem is of no astronomical value unless it is adapted to practical *numerical computation*, and will give accurate numerical results, from data of the character usually available, without the expenditure of undue time and labor.

A highly successful method for actually accomplishing the calculation of the orbital elements from observed positions in the sky was

devised for comets by Olbers in 1797, and is still much used; but most comets travel in such elongated (highly eccentric) ellipses that the small part of the orbit over which they are visible does not usually depart much from a parabola, and Olbers' method is based on the explicit hypothesis that the orbit is parabolic, and hence cannot be used for a planet, nor even for all comets. Laplace, in 1780, suggested a method for accomplishing the calculation without any hypothesis about the form of the orbit, but it was doubted at the time whether his method would work satisfactorily in practice, and since no occasion to use it had arisen, it was not developed and applied. Upon the discovery of Ceres, the first of the minor planets, however, astronomers were suddenly confronted with the unprecedented problem of determining accurately the orbit of a planet that had been under observation for only a few weeks, because before many observations had been obtained Ceres became invisible in the evening twilight, and unless its position several months later could be calculated it probably could not again be found among the myriads of the stars. Gauss, then only 24 years old, attacked the problem; and succeeded in producing such a satisfactory solution that his method, with later improvements and modifications by Encke and others, is still in extensive use. In recent years, however, Laplace's method has also been modified and developed, especially by Leuschner, into a convenient form now widely employed.

Laplace's method is based directly on the above equations of motion; by neglecting the mass of the body (which is unknown at the start, anyway) in comparison with that of the sun, and taking $1/k = 58.13244$ days as the unit of time, these equations may be written in the form

$$\frac{d^2x}{dt^2} = -\frac{x}{r^3}, \quad \frac{d^2y}{dt^2} = -\frac{y}{r^3}, \quad \frac{d^2z}{dt^2} = -\frac{z}{r^3},$$

in which the heliocentric rectangular coordinates x,y,z are referred to right handed Cartesian axes with origin at the sum, XY-plane in the plane of the equator, $X-$axis directed toward the vernal equinox and $Z-$axis toward the north celestial pole, and specify the position of the body relative to the sun. The position of the sun is specified by its coordinates X,Y,Z, relative to the earth, referred to Cartesian axes parallel to the preceding ones, but with origin at the earth; the geocentric rectangular coordinates ξ,η,ζ of the body referred to the latter axes are then

$$\xi = x+X, \qquad \eta = y+Y, \qquad \zeta = z+Z.$$

Then after certain small corrections to allow for the fact that the observer is not at the center of the earth, the right ascension α and the

declination δ, which are measured by observation and specify the position in the sky, are, with the geocentric distance ρ, the corresponding spherical coordinates, and hence are connected with the rectangular coordinates by the relations

$$\xi=\rho\lambda, \qquad \eta=\rho\mu, \qquad \zeta=\rho\nu,$$

in which $\quad \lambda=\cos\delta\cos\alpha, \quad \mu=\cos\delta\sin\alpha, \quad \nu=\sin\alpha,$

are the direction cosines of the object as seen from the earth.

Now, substituting $x=\xi-X=\rho\lambda-X$, etc., in the equations of motion gives

$$\frac{d^2(\rho\lambda)}{dt^2}-\frac{d^2X}{dt^2}=-\frac{\rho\lambda-X}{r^3}, \quad \text{etc.};$$

but the coordinates of the sun relative to the earth are merely the negatives of those of the earth relative to the sun, and since the earth itself (aside from perturbations) also satisfies the above equations of motion, we have

$$\frac{d^2X}{dt^2}=-\frac{X}{R^3}, \quad \text{etc.},$$

where R is the distance of the sun from the earth. Making this substitution, we have the fundamental equations of the Laplacian method,

$$\frac{d^2(\rho\lambda)}{dt^2}+\frac{\rho\lambda}{r^3}=X\left(\frac{1}{r^3}-\frac{1}{R^3}\right), \quad \text{etc.}$$

Let t_1,t_2,t_3 be the three times at which the right ascension and declination have been observed; then at time t_2

$$\lambda_2\rho''_2+2\lambda'_2\rho'_2+\left(\lambda''_2+\frac{\lambda_2}{r_2^3}\right)\rho_2=-X_2\left(\frac{1}{R_2^3}-\frac{1}{r_2^3}\right), \quad \text{etc.},$$

and $\qquad r_2^2=\rho_2^2+R_2^2-2\rho_2R_2\cos\psi_2,$

in which primes denote differentiations, and the last equation comes from the Law of Cosines applied to the triangle formed by the sun, the earth and the planet, with the angle at the earth denoted by ψ. These four equations contain only four unknowns: r_2, ρ_2, ρ_2', ρ_2''. Of the other quantities, λ_2, μ_2, ν_2 are given immediately by α, δ; and the values of their first and second derivatives at t_2 can be obtained with sufficient accuracy from the three observed values (e. g., by numerical differentiation by Lagrange's interpolation formula, or from the Taylor expansions of α, δ, about t_2), while X,Y,Z,R are given in the

Ephemerides published annually by the principal governments of the world (interpolation is necessary, e. g. by Bessel's Formula); finally, from elementary analytic geometry $R\cos\psi=\lambda X+\mu Y+\nu Z$. These equations can therefore be solved simultaneously for the unknowns: The first three equations are linear in ρ_2, ρ_2', ρ''_2, whence ρ_2 and ρ'_2 are easily found in terms of r_2 (ρ'' is not needed); the result for ρ_2 solved simultaneously with the fourth equation gives the actual values of ρ_2 (through the solution of an eighth degree algebraic equation, known as Gauss' equation, which arises in the elimination of r_2) and r_2. With ρ_2, ρ_2' known, we have

$$x_2=\rho_2\lambda_2-X_2, \qquad x_2'=\rho_2\lambda_2'+\rho_2'\lambda_2-X_2', \quad \text{etc.},$$

X_2', Y_2', Z_2' being found by numerical differentiation from the Ephemeris. From these coordinates and velocities at t_2, the orbital elements are computed by means of the relations between the elements and the constants of integration in terms of initial conditions.

The intermediate elements in the Laplacian method are the coordinates and their derivatives (position and velocity) at the time of the middle observation; in Gauss' method, they are the coordinates (positions) at the times of the first and the third observations:

Since the orbit is a conic section, it lies in a plane through the sun. The condition that the plane of the orbit $Ax+By+Cz=0$ pass through the three observed positions is that

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=0,$$

which developed by minors with respect to each column in succession gives

$$c_1x_1-x_2+c_3x_3=0,$$

$$c_1y_1-y_2+c_3y_3=0,$$

$$c_1z_1-z_2+c_3z_3=0,$$

where $$c_1\equiv\frac{[23]}{[13]}, \qquad c_3\equiv\frac{[12]}{[13]}$$

are the ratios of twice the areas of the triangles [23], etc., formed by the sun and the three positions of the planet taken in pairs. Integrating the equations of motion in powers of $t-t_2$ gives

$$x=fx_2+gx'_2, \qquad y=fy_2+gy'_2, \qquad z=fz_2+gz'_2,$$

in which f, g, are known power series in $t-t_2$ with coefficients in r_2 and its derivatives, from which the coordinates and hence the ratios of the triangles c_1, c_3 may be expressed as power series in the time intervals between the observations. Substituting these series for the ratios, and substituting $x=\rho\lambda-X$, etc., into the above equations gives the fundamental equations of the Gaussian method:

$$\lambda_1(c_1\rho_1)-\lambda_2\rho_2+\lambda_3(c_3\rho_3)=X_1c_1-X_2+X_3c_3, \quad \text{etc.}$$

These equations, with the same triangle equation as before, again form four simultaneous equations in four unknowns ρ_1, ρ_2, ρ_3, r_2. The first three are linear in ρ_1, ρ_2, ρ_3, and may be solved for these quantities in terms of r_2; from the result for ρ_2 and the triangle equation, ρ_2 is found as before, through the same eighth degree equation, whence r_2, c_1, c_3, ρ_1, ρ_3 become known. From ρ_1, ρ_3, the six coordinates at t_1, t_3, viz.,

$$x_1=\rho_1\lambda_1-X_1, \ldots\ldots, \qquad x_3=\rho_3\lambda_3-X_3, \ldots\ldots,$$

are found; and from them the orbital elements may be computed.

It will be noted that the determination of the orbit in which a newly discovered object is moving is exceedingly simple in principle; but the actual computations are lengthy and laborious, and require a somewhat elaborate array of formulæ. Many different variants of the two fundamental procedures sketched above have been developed from time to time; and numerous different sets of working formulæ for putting them into practice have been devised, both for logarithmic calculation and for machine computation.*

In practice, the application of the above methods usually requires a series of successive approximations, because the formulæ involve infinite series in which higher terms are neglected and hence the conic first obtained does not in general pass exactly through all three of the observed positions. The final approximation, which satisfies the three observations (or more, in the exceptional cases when more than three are necessary) is not itself likely to be highly accurate, however, although it serves to predict the motion more or less closely for a limited period in the immediate future: *Short* time-intervals between the observations—from one or two to several days—are essential (both to quickly obtain a provisional orbit as soon after discovery as possible, and to make the formulæ as efficient as possible); and the inevitable small ***errors of observation*** which are present in all scientific measure-

*The arrangements of formulæ now in common use will be found in: R. T. Crawford, *Determination of Orbits of Comets and Asteroids*, New York, 1930; K. P. Williams, *Calculation of Orbits of Asteroids and Comets*, The Principia Press (Bloomington, Ind.) 1934; G. Stracke, *Bahnbestimmung der Planeten und Kometen*, Berlin 1929.

ments may lead to comparatively large errors in the orbit computed from observations that include such a small arc of the curve. Corrections must therefore be subsequently applied to the preliminary orbit on the basis of the differences between the positions of the body given by further observations and the positions computed from the provisional orbit. Even in the absence of exceptional circumstances, the determination of an orbit sufficiently accurate for the certain recovery of a minor planet at only its next apparition requires in general a series of good observations over an interval of several weeks; the derivation of this accurate orbit from long arcs requires the approximate orbit to be already known. A *definitive* orbit, not subject to further correction, is not possible until observations have been obtained over an interval of several complete revolutions at points well distributed around the orbit; and is determined by least-squares corrections to the previous values of the elements. A large proportion of the newly discovered minor planets become lost before enough observations are secured to obtain reliable orbits.

With the orbital elements known, the position of a planet in its orbit at any given past or future time is easily calculated in the case of Keplerian motion by simple formulæ derived from Kepler's laws; the position in space, and finally the location on the celestial sphere as seen from the earth, are then readily obtained by elementary geometry and trigonometry, and the data in the Ephemeris. It must be remembered, however, that even though the orbit has been determined with high precision, it still is necessary to determine the perturbations, or departures from Keplerian motion which are produced by the attractions of bodies other than the sun, in order to calculate the actual motion accurately over any very long interval of time. This problem is an intricate one, and cannot here be discussed in detail; but the irregularities thus introduced into the celestial motions are of fascinating interest, and the reader who wishes a simple introduction to them is referred to the excellent non-mathematical account of their principal features given by Sir John Herschel in Part II of his *Outlines of Astronomy*. Numerical integration of the differential equations of disturbed motion is extensively used in the actual calculations of perturbations; but in many cases analytical methods are also employed, among which is the familiar method of *variation of parameters*.

A Note on the 3-Bar Curve

By Robert C. Yates
Louisiana State University

It is the purpose of this note to call attention to a fact in connection with the much discussed 3-bar curve which, although somewhat obvious, seems so far to have been overlooked.

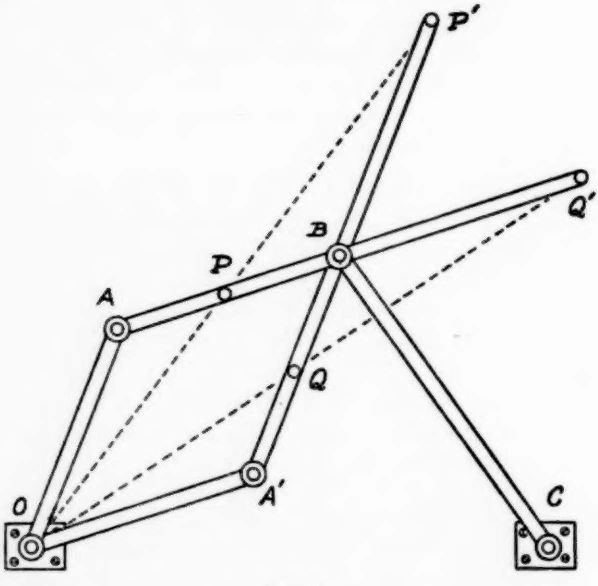

Fig. 1.

In the appartus of Fig. 1 the point P traces the 3-bar curve. It is well known that if the radial bar OA and the traversing bar AB be interchanged then an exactly similar curve is described.* This last is the mechanism OA', $A'P'$, BC, with P' as the tracing point. The principle is self-evident when one recognizes the parallelogram linkage $OAA'BP'$ as the ordinary pantograph, with O, P, and P' collinear.

However, if the sides OA and AB of the parallelogram $OAA'B$ be taken equal in length, then the points P and Q, equidistant from B,

*First noticed by A. B. Kempe who was thus led to the quadruplane and skew pantograph. See Sylvester: *The Plagiograph aliter the Skew Pantigraph*, Nature XII, (1875) pp. 168, 214-216.

describe the same curve. This too is obvious when one notices that OA and AB are interchangeable with OA' and $A'B$ for a fixed position of the bar BC.

It follows at once that the point Q', attached to the bar AB so that it is collinear with O and Q, describes the same curve as does P'. Thus we have double generation of the two similar 3-bar curves.

It may be of interest to list a few of the many striking properties of this famous curve which is traced in general by a point of a traversing *plate* attached to two radial bars:

1. The curve is a sextic which may degenerate, for example, into a circle and a lemniscate or into three lines.

2. Any particular 3-bar curve may be generated by a triple 3-bar mechanism. This remarkable characteristic may be put in different

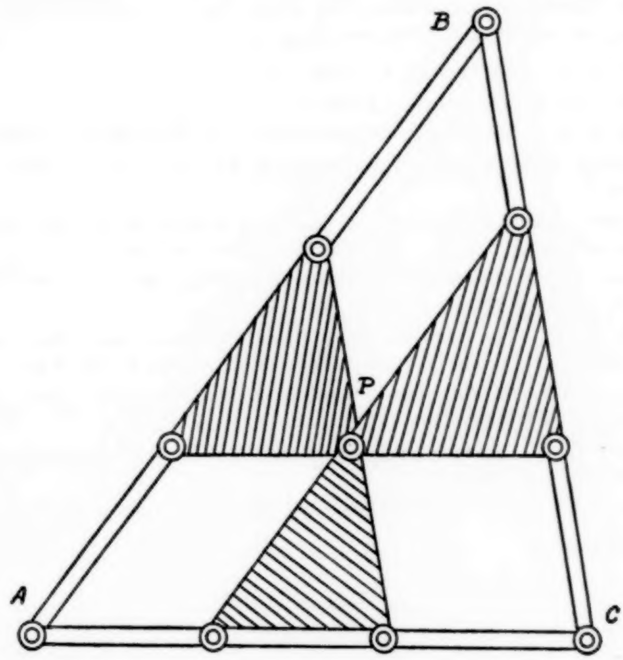

FIG. 2.

light: Build a linkwork as in Fig. 2 where A,B,C are arbitrarily selected points and P, the tracer, is the intersection of lines drawn parallel to the sides of triangle ABC. Then no matter how this linkwork is deformed, triangle ABC remains at all times similar to its original form. Thus if A and C be fixed in some position then P describes the

curve and the point B remains at rest. Or, if all three points A,B,C be fixed, all three of the 3-bar mechanisms produce the same curve in mutual cooperation.

3. The curve may be transformed into itself by means of an isogonal transformation whose defining triangle is determined by the double points of the curve.

4. The points A,B,C (see Fig. 2) and the three double points of the curve have the same sum (mod 2π) for their central vectorial angles.

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*Not exhaustive.

A Formula for $\sum_{x=1}^{x=n} x^p$

By J. Rosenbaum
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It is the purpose here to show the derivation of a formula for the sum of the pth powers of the first n integers, where p is a positive integer.

In what follows, the notation n_r will be used to denote the rth coefficient in the expansion of $(a+b)^n$, where n is a positive integer or zero.

It is noted that this notation is a substitute for the familiar combination symbol, i. e. $n_r = {}_nC_{r-1}$.

For future reference, we write down the known formula:

$$(n+1)_{r+1} = n_{r+1} + n_r. \tag{1}$$

We also state the definition:

$$n_r = 0, \text{ when } r<1, \text{ and also when } r>n+1. \tag{2}$$

It is now proved by induction:

$$\begin{aligned} f(n,r) = n_r - (r-1)_2 n_{r+1} + r_3 n_{r+2} - (r+1)_4\, n_{r+3} + - + \cdots \\ +(-1)^{n+r}\,(n-2)_{n+1-r}\, n_n + (-1)^{n+r+1}\,(n-1)_{n+2-r}\, n_{n+1} = 1, \end{aligned} \tag{3}$$

where n and r are integers such that $1<r<(n+2)$.

Proof: By (1),

$$\begin{aligned} f[(k+1),\, r] = (k_{r-1}+k_r) - (r-1)_2(k_r+k_{r+1}) \\ + r_3(k_{r+1}+k_{r+2}) - (r+1)_4(k_{r+2}+k_{r+3}) + - + \cdots \\ +(-1)^{k+r+1}\,(k-1)_{k+2-r}\,(k_k+k_{k+1}) \\ +(-1)^{k+r+2}\,k_{k+3-r}\,(k_{k+1}+k_{k+2}). \end{aligned} \tag{4}$$

Collecting like terms, and observing that by (2) the second term in the last parenthesis equals zero, (4) can be written as

$$\begin{aligned} f[(k+1),\, r] = k_{r-1} - [(r-1)_2 - 1]k_r \\ + [r_3 - (r-1)_2]k_{r+1} - [(r+1)_4 - r_3]k_{r+2} + - + \cdots \\ +(-1)^{k+r+1}\,[(k-1)_{k+2-r} - (k-2)_{k+1-r}]k_k \\ +(-1)^{k+r+2}\,[k_{k+3-r} - (k-1)_{k+2-r}]k_{k+1} \end{aligned} \tag{5}$$

By virtue of (1), and because the exponents of (-1) can each be decreased by 2, (5) is the same as

$$f[(k+1),\, r]=f[k,\,(r-1)]. \tag{6}$$

Equation (6) states that for all values of r in the range specified under (3), except perhaps for the value $r=2$, if (3) holds for *one* value of n such as k, then it also holds for $n=k+1$. However, when $r=2$ the right of (6) equals 1, by virtue of (2). So, the possible exception $r=2$, is removed. Furthermore, as regards the upper limit of r as specified under (3) it is seen that if $(r-1)<k+2$ then $r<(k+1)+2$. Hence if (3) holds for $n=k$ with the specified range for r, then it also holds for $n=k+1$, with the corresponding range for r.

This together with the fact that (3) holds for $n=1$ completes the proof.

From (3) it follows that if a_i, $i=1,2,\cdots,n$, are any n numbers then

$$\sum_{i=1}^{i=n} a_i = \sum_{i=1}^{i=n} a_i f[n,(i+1)]. \tag{7}$$

By rearranging the terms on the right of (7) there is obtained the following equation which forms the basis of the formula in question:

$$\begin{aligned}(8)\qquad \sum_{i=1}^{i=n} a_i &= a_1 n_2+(a_2-a_1)n_3+(a_3-2a_2+a_1)n_4+\cdots\\ &+[a_{r-1}-(r-2)_2\, a_{r-2}+(r-2)_3\, a_{r-3}-+-\cdots+(-1)^r a_1]\, n_r+\cdots\\ &+[a_n-(n-1)_2\, a_{n-1}+(n-1)_3\, a_{n-2}-+-\cdots+(-1)^{n+1}a_1]\, n_{n+1}.\end{aligned}$$

It is clear that if i^p, where p is a positive integer, be substituted for a_i in (8), then the left of the resulting equation becomes the very sum for which a formula is here sought. However, the resulting equation is, without further consideration, not a *formula* in the ordinary sense, because apparently it seems that when the number of terms on the left increases then the number of terms on the right also increases. That this is not the case (that is, that when a_i are the numbers mentioned above, then for every fixed p, the number of terms on the right of (8) is limited, regardless of how large n may be), is shown by the following:

Theorem. If p is a positive integer or zero, and if n is an integer *greater* than p, and if a is any number, then

$$\begin{aligned}(9)\qquad g(n,p,a)=a^p-n_2(a+1)^p+n_3(a+2)^p-+-\cdots\\ +(-1)^n(a+n)^p=0.\end{aligned}$$

This is proved by induction, thus:

Let k be one value of n for which (9) holds, with the corresponding restriction on p. That is, let

$$g(k,p,a)=0,\quad 0\leqq p<k. \tag{10}$$

On account of the absence of restrictions on a, (10) can be written as

$$g[k,\ p,\ (a+1)]=0,\quad 0\leqq p<k. \tag{11}$$

The subtraction of (11) from (10), with a view of (1), yields

$$g[(k+1),\ p,\ a]=0,\quad 0\leqq p<k. \tag{12}$$

It is observed that (12) does not complete the chief argument in the proof, because, so far, it has been proved that (12) is true for every p less than k, but not for every p less than $k+1$. To remove this remaining restriction on p, we first observe that

$$(k+1)g(k,p,1)=-g[(k+1),(p+1),\ 0]. \tag{13}$$

From (10) and (13),

$$g[(k+1),(p+1),\ 0]=0,\quad 0\leqq p<k. \tag{14}$$

Next, by collecting like powers of a, it is found that

$$\begin{aligned} g[(k+1),(p+1),\ a]&=g[(k+1),0,1]a^{p+1}\\ &+g[(k+1),1,0](p+1)_2a^p+g[(k+1),2,0](p+1)_3a^{p-1}+\cdots\\ &+g[(k+1),r,0](p+1)_{r+1}a^{p-r+1}+\cdots\\ &+g[(k+1),p,0](p+1)_{p+1}a+g[(k+1),(p+1),0]. \end{aligned} \tag{15}$$

Now if $p<k$ then by (12) every term on the right of (15) but the last, is known to vanish, and by (14) every term but the first is known to vanish. Hence

$$g[(k+1),(p+1),a]=0,\quad 0\leqq p<k. \tag{16}$$

This is equivalent to

$$g[(k+1),p,a]=0,\quad 0<p<(k+1). \tag{17}$$

Furthermore, it is well known that (9), and hence also (17) is true when $p=0$. This, together with the fact that (9) holds for $n=1$, completes the proof.

Now if in (8), i^p is substituted for a, then the coefficient of n_r becomes

$$(-1)^r g[(r-2),p,1].$$

Hence, by (9), this coefficient vanishes for every r greater than $(p+2)$, and the resulting equation becomes

$$(18)\quad \sum_{x=1}^{x=n} x^p = n_2 + (2^p-1)n_3 + (3^p - 2\cdot 2^p + 1)n_4$$
$$+ (4^p - 3\cdot 3^p + 3\cdot 2^p - 1)n_5 + \cdots$$
$$+ [(p+1)^p - p_2\cdot p^p + p_3(p-1)^p - p_4(p-2)^p + - + \cdots$$
$$+ (-1)^p]n_{p+2}.$$

This is the formula sought.

The question naturally arises as to the impelling forces which lead to the study of subjects like algebra and to the "clothed problem" solved by means of equations. What human needs were satisfied by subjects like these, and what kept the latter throughout the ages?

The answer is "the puzzle," which has always been a moving force in all branches of learning. The ancients were puzzled by the movements of the stars, and so are the moderns of our day. To solve these puzzles is at present the work of astronomy. The world is puzzled as to why we are here, whence we came, and where—if anywhere—we are to go, and so religion comes to our assistance. The general biquadratic equation puzzled the world for centuries, and it was the puzzle-solving instinct that led Ferrari, about 1545, to find the solution. It is the joy of the puzzle which leads most pupils to solve a set of simultaneous equations and the lawyer to solve the legal problems which a new case proposes. All these are human intellectual puzzles and all will continue to be such so long as human beings are human.—From David Eugene Smith in his *The Story of Mathematics*, as related in Scripta Mathematica Forum Lectures, published by Yeshiva College.

Humanism and History of Mathematics

Edited by
G. Waldo Dunnington

A History of American Mathematical Journals

By Benjamin F. Finkel
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INTRODUCTION

As one might expect, little progress was made in the study of mathematics in America from the time of its first permanent settlement to the beginning of the nineteenth century. The early settlers of America, having, for the most part, fled from Europe to escape religious persecution were, naturally, most concerned with matters of religion. Hence, what time they had to spare from resisting the attacks of hostile Indians on the one hand, and making homes and developing the natural resources of the most wonderful country in the world, on the other, was devoted to matters of religion and government. No time was given to the study ot the natural sciences or to mathematics. In fact, the religious restrictions imposed for many years after the founding of New England was such as to stifle investigation and smother genius. However, while no attention was given to the study of mathematics by the colonists, much importance was attached to the study of languages in general and Greek and Latin in particular. Thus, many of the early settlers, especially the Clergy, were among the most profound classical scholars in the world. We need only to recall the scholarly attainments of Cotton and Increase Mather, and later of Jonathan Edwards, to remind us that in linguistic attainments and metaphysical speculations, they were the equal of any of their contemporaries in Europe. A similar statement regarding mathematics cannot, however, be made. But, in comparing the progress made in the study of mathematics in America with that in Europe, one must not forget that the mathematical bud had just begun to open in Europe when the first permanent settlements were made in America. Thus, Descartes had not yet contributed to mathe-

matical science Analytical Geometry; Fermat had not then made his profound discoveries in the Theory of Numbers; and Newton and Leibnitz had not discovered the Infinitesimal Calculus. For this reason, one must not compare on equal footing, the low state of mathematical learning in a nation just struggling into existence with that of nations whose history stretches back through a score of centuries. The study of mathematics requires the power of prolonged and concentrated effort without interruption. Certainly, the trying events of the early days of American colonization were not conducive to the quiet, contemplative attitude of mind, which invents an Infinitesimal Calculus, writes a *Méchanique Céleste*, or produces a *Disquisitiones Arithmeticæ*. Jefferson's apt reply to the assertion of Abbé Reynal that America had not yet produced one good poet, one able mathematician, one man of genius in a single art or a single science was pertinent when he said, "When America shall have existed as a people as long as the Greeks did before they produced a Homer, the Romans a Virgil, the French a Racine and Voltaire, the English a Shakespeare and Milton, should this reproach still be true, we will inquire from what unfriendly causes it has arisen that the other countries of Europe and quarters of the earth shall not have inscribed any names in the roll of poets..... In war, we have Washington whose memory will be adored while liberty shall have votaries, whose name shall triumph over time and will in future ages assume its just station among the most celebrated worthies of the world.... In physics, we have a Franklin than whom no one of the present age has made more important discoveries, or more ingenious solutions of the phenomena of nature. We have supposed Mr. Rittenhouse second to no astronomer living; that in genius he must be first because he is self taught.... He has indeed not made a new world; but he has by initiation approached nearer its Maker than any man who has lived from the creation to this day."*

It is not a matter of surprise to find that even Arithmetic was not taught in America until nearly 150 years after the landing of the Pilgrims. "Thus, in Hampstead, New Hampshire, in 1750, it was voted 'to hire a schoolmaster for six months in ye summer season to teach ye children to read and write'."† Without commerce the need of the study of Arithmetic was not imperative. Even at the beginning of the eighteenth century there were rural schools in which arithmetic was not taught at all, and there were many schools at the beginning of the twentieth century, and there are many schools now, in which the teaching of arithmetic is worse than if it were not taught

*Jefferson's *Notes on Virginia*. Quoted in *The Teaching and History of Mathematics in the United States*, by Florian Cajori.

†Cajori. *Ibid.*, p. 41.

at all. The reason for this is that it is too often taught by teachers who are ignorant of its fundamental principles, who have no sympathy with it and in many cases, a positive aversion for it. How can such teachers assume the mental attitude of the child and establish a connection between the ideas that the child already has and those which the teacher should impart? As an evidence of the lack of knowledge and preparation on the part of teachers of Arithmetic, we give the following incident: In the summer of 1905, we chanced to step into the Capitol of one of the greatest states in the Union at the time. The State Teachers' Association was in session. Pausing for a moment to get the speaker's line of argument, we heard him propose this problem to his audience: Suppose we have a square piece of land whose sides are 8 rods. Inclose this tract with a wire fence, the posts of which are to be a rod part. Also connect the middle points of the opposite sides of the tract with a wire fence, the posts of which shall be a rod apart. How many posts will it take? While the problem was proposed with a view of testing the power of imagination of his auditors, his subject being *The Cultivation of the Imagination*, many used pencil and paper and out of the fifteen answers publicly announced, only *one* was correct. Yet these were teachers "teaching the young ideas how to shoot"!

The method of presentation of arithmetic has much to do in arousing interest in its study. The early teaching of Arithmetic consisted in the use of Roman numerals which were taught in connection with other elementary notions, the text used being a sheet mounted on wood and protected by transparent horn, the apparatus being called the "hornbook".

The first purely arithmetical textbook used in America was the arithmetic of James Hodder, a famous English teacher of the 17th century. The first purely arithmetical book known to have been printed in this country was an American edition printed from the 24th English edition of Hodder's. The first arithmetic by an American author and printed in America was that of Professor Isaac Greenwood, of Harvard College, 1779. Professor Cajori says, "So far as I know, only three copies of this book are extant, two in the Library of Harvard College, and one in the Congressional Library". Dillworth's *Arithmetic*, published in London, about 1744, was the first to reach an extended circulation in the colonies. This book contained "a short collection of pleasant and diverting questions", such, for example, as the fox, the goose, and the peck of corn,—a question which has interested several hundred generations. In the edition of 1798, a copy of which is in my possession, there are nine of these questions occupy-

ing page 180. It aslo contains a picture of the author, Thomas Dillworth. It is likely that the questions referred to had much to do with the popularity of the book. Copies of this book are scarce.

Of the teaching of mathematics in our oldest institutions of learning in the early days, little is known. At Harvard, the oldest of our colleges, the courses of study in mathematics were very elementary for many years after the founding of the college. Thus, in 1643, a student was not required to take an entrance examination in mathematics. The principal requirements for admission were Latin and Greek. The requirements for admission in Latin were, however, such as would put to grief and shame a very large percentage of the 20th century candidates for admission to college. At that time, a candidate could be ignorant of the four fundamental processes of Arithmetic. Happily for the mighty progress made in all branches of learning even in spite of such deadening influences, these conditions have greatly changed.

An examination of the courses of study at Harvard in its early days shows that mathematics was not taught at all except during the third year of the college course. The course in mathematics began in the senior year and consisted of Arithmetic and Geometry during the first three quarters and Astronomy during the last quarter. The importance attached to mathematics as compared with other branches of study may be inferred by the time devoted to each subject. Thus, ten hours were devoted to Philosophy, seven to Greek, six to Rhetoric, four to Oriental languages, while only two were devoted to mathematics. The mathematical course at Harvard remained apparently as above until the beginning of the 18th Century.

Algebra was introduced at Harvard sometime between 1726 and 1738. This came about by the adoption of Ward's *Mathematics*, a book divided into five parts. The first part was devoted to Arithmetic; the second part to Algebra; the third part Geometry; the fourth part to Conic Sections; and the fifth part to the Arithmetic of Infinities. This text was used for many years at Harvard, Yale, Pennsylvania, Brown, and Dartmouth. By the beginning of the 19th Century, some provisions were made at Harvard for the study of the Calculus, then called Fluxions, as appears from the theses written by Juniors and Seniors between 1781 and 1807 and deposited in the Harvard Library.

It is very certain that progress in the study of mathematics at Harvard would have been more rapid had the authorities of the college not adopted the paralyzing method of providing for mathematical instruction. As appears from a diary of a student who was at Harvard

in 1786, complaints were made that the Greek tutor was too young. "Before he took his second degree which was last Commencement, he was chosen tutor of mathematics". This pernicious practice of giving preference for every occupation to graduates who excel mainly in Greek and Latin is still quite common in many of the smaller colleges where the classics are held to be the *summum bonum* in education. Of course, there is still an easy psychological explanation why such a practice still survives. Professor John Perry, of England, has very forcibly set forth the reasons why classical instructors are so ready and eager to recommend and urge for promotion to all sorts of positions, classical graduates. A partial reason from a student's stand-point may be seen by the following quotation in the Harvard Lyceum started in 1810 and in which there was, at that time, an effort made to arouse interest in mathematics among the students: "Perhaps, no science has been so universally decried by the overwhelmingly dull as the mathematics. Superficial dabblers in science, contented to float in doubts and chimeras, and unable to see the advantage of demonstrable truth, turn back before they have passed the narrow path which leads to the firm ground of mathematical certainty, and not willing to have others more successful than themselves, like the Jewish spies, they endeavor to deter them from the way by horrid stories of giant spectres in the promised land of demonstration, and scarcely a Caleb is found to render a true account of its beauties". This same reason accounts largely for the lamentable condition of the teaching of mathematics in our high schools, academies, and colleges. Fortunately, the advocates and devotees of the almost exclusive classical education are rapidly diminishing in number, and, perhaps, in a few generations more, the last of that peculiar educational genus, *viz.*, the men who hold that a student's time is profitably employed only when it is devoted to the study of the classics, will have become extinct.

At Yale, the earlier advantages for mathematical instruction fared no better than at Harvard. Thus, says Cajori: "During the first seventeen years at Yale, the doctrine of the schoolmen in logic, metaphysics, and ethics, still held sway. Descartes, Boyle, Locke, Bacon, and Newton were innovators from whom no good could be expected. It is pleasing to think that the introduction of Newtonian ideas and the rise of mathematical studies at Yale was partly due to an act of charity by the great Newton himself." However, it appears that Yale had made provisions for the study of Fluxions previous to 1766, thus anticipating Harvard in this important step by several years.

The mathematical course in the University of Pennsylvania in 1758, as given by William Smith, D. D., its first provost, was the equal,

if not superior to that offered by any other institution in America. How closely the course was followed, is not known.

The mathematical text-books used in America prior to the 19th Century, and even for some time after the 19th Century were chiefly American editions of English works. We have already mentioned the first arithmetic used in the Colonies. In geometry, the standard was, of course, translations from the Greek geometry of Euclid. The first American edition, it seems, appeared at Worcester in 1784. In 1803, Thomas and George Palmer, in Philadelphia, published Robert Simpson's *Euclid*, together with the *Elements of Plane and Spherical Trigonometry*. Copies of this work are yet quite numerous. Other editions of Euclid were brought out by John D. Craig, Baltimore, 1818; Rev. John Allen, Professor of Mathematics in the University of Maryland, 1822; and Playfair's Euclid by F. Nicholls in 1806.

The first Algebra published in this country was an edition of Simpson's *Algebra*, 1809. In 1822, James Ryan revised and edited Bonnycastle's *Introduction to Algebra*. The first Calculus published in America was an American edition of Rev. S. Vince's *The Principles of Fluxions*, 1812.

The first Calculus written and published by an American author was *The Differential and Integral Calculus*, by James Ryan, 1828. The writer possesses a copy of this very rare work.

The first periodical published in the United States and exclusively devoted to mathematics was established in 1804. To the history of Mathematical Periodicals and Journals published in the United States, we shall now direct our attention.

JOURNALS OF AMERICA DEVOTED EXCLUSIVELY TO MATHEMATICS

From the brief survey of the study of mathematics in America which we have just made, we observe that at the beginning of the 19th Century, a manifest interest in the study of mathematics begins to appear.

About this time, an organization was formed and called the New York Mathematical Club, the object of the organization being the cultivation and dissemination of mathematical thought among its members. As a result of this organization, it was resolved to establish a mathematical periodical which should do for the study of mathematics in America what Leybourn's *Ladies Diary* had, for a century, been doing in England. The periodical established was styled *The Mathematical Correspondent*. It was to contain only one sheet, or 24 pages, and was to be published quarterly. Several editors were ap-

pointed of whom George Baron, an irascible and egotistic character, was editor-in-chief. It does not appear who his associate editors were or how many there were.

The first number of the *Correspondent* was issued in New York, May 1, 1804. Nine numbers, the ninth, however, being a double number, were published and these constituted Volume I. The double "Numbers 9, 10, etc.", as stated on the outside of the front cover, was issued November 18, 1806. With this number, George Baron's connection with the *Correspondent* ceased, owing, as he informs us in a note "To Subscribers", which is on the inside of the front cover of double "Numbers 9, 10, etc.", to his intention to publish a "large work of great importance to seamen and others". With the loss of Mr. Baron's support, the associate editors abandoned the enterprise they had almost completely wrecked, because they had lost the respect of gentlemen and the support of their subscribers, through the contemptuous way in which not only some of the contributors but the editors themselves, spoke of the works and contributions of many of their contemporaries. As an illustration of the manner in which these editors were wont to speak of some of the most noted and authentic works of their time, we quote what David S. Hart of Stonington, Connecticut, says is on the cover of Number 2, Volume I, a copy of which he had in his possession in 1875. According to Dr. Hart, on the cover of Number 2 is an advertisement of a lecture delivered in New York by G. Baron, which as Baron says, contains a complete refutation of the false and spurious principles, ignorantly imposed on the public in the *New American Practical Navigator* written by N. Bowditch and published by G. W. Blunt. The sub-editors say, "We agree with the author that he has shown in the most incontrovertible manner, that the principles on which the *New American Practical Navigator* is founded, are universally false, and gross imputations on the public". This work was the most authentic and trustworthy text that had been written up to the time of its appearance. Our copy, 18th Edition, contains 776 pages and is a veritable encyclopedia of navigation facts.

The following is the title page of the *Mathematical Correspondent:*

THE
MATHEMATICAL CORRESPONDENT

Containing

NEW ELUCIDATIONS, DISCOVERIES,
AND IMPROVEMENTS,
IN VARIOUS BRANCHES OF THE MATHEMATICS;

with

COLLECTIONS OF MATHEMATICAL QUESTIONS RESOLVED
BY INGENIOUS CORRESPONDENTS.

Adapted

TO THE PRESENT STATE OF LEARNING IN AMERICA.

And Designed,

TO INSPIRE YOUTH WITH THE LOVE OF MATHEMATICAL KNOWLEDGE,
BY ALLURING THEIR ATTENTIONS TO THE SOLUTIONS
OF PLEASANT AND CURIOUS QUESTIONS—AND TO PROMOTE
THE CULTIVATION OF THE MATHEMATICS,
BY OPENING A CHANNEL FOR THE READY CONVEYANCE
OF DISCOVERIES AND IMPROVEMENTS,
FROM ONE MATHEMATICIAN TO ANOTHER.

VOL. I.

"In the Mathematical Sciences, truth appears most conspicuous, and shines in its greatest lustre."—EMERSON.

NEW YORK

PRINTED BY SAGE AND CLOUGH,
For the Editors, and Sold by T. and J. Swords,
No. 160, W. Falconer, No. 112, Sage and Thompson, No. 149, Pearl Street, and G. Baron, No.
79 Pine Street.
1804.

The aim and purpose of the *Mathematical Correspondent* may best be gained from the Preface, which reads as follows: "When we consider the great exertions of learned men to disseminate mathematical information in other countries, we must be surprised to find that this

kind of knowledge is most shamefully neglected in the United States of America. The mathematical sciences are the foundations of almost every art that is necessary to promote the comfort and convenience of civilized man; their extensive use in human affairs stands attested by the wise and learned of every age. It cannot, therefore, be denied that these sciences ought to be diligently studied and liberally encouraged in a country like this. But these sciences are as valuable in their own nature as they are useful. For, by mathematical exercise, as the celebrated Dr. Barrow observes, the mind is inured to a constant diligence in study, delivered from a credulous simplicity, strongly fortified against the vanity of skepticism, restrained from rash presumption, inclined to due ascent, subject to the government of right reason, and inspired with resolutions to combat the unjust tyranny of false prejudices. In Europe, small periodical publications have contributed much to the diffusion of this kind of learning, and many of the greatest scientific characters of the present age have at an early age of life commenced their mathematical career by answering the questions proposed in such works. Ambitious of seeing their little productions in print, they study with an ardor which nothing else could inspire; and this ambition properly encouraged, by a judicious parent or preceptor, accelerated the improvement of the pupil and formed the basis of future eminence. The learned Dr. Hutton in a late publication has declared that a small periodical work, entitled the *Ladies' Diary,* had produced more mathematicians in England than had been done by all the mathematical authors in the Kingdom.

But this is not the only advantage that has resulted from such works. They have served as useful vehicles in conveying ideas, discoveries, and improvements from one mathematician to another and to the community at large. By these means, the limits of the sciences have been extended and the common stock of mathematical knowledge continually augmented and extended. The *Mathematical Correspondent* will be conducted on the same plan as the European works to which we have just alluded; as similar causes generally produce similar effects, we are not without hope of rendering some service to the public.

A number of this work containing one sheet of paper, will be regularly published four times a year, viz. on the first days of May, August, November, and February. In each number a prize question will be proposed and whoever gives the best solution to that question one month previous to the publication of the next succeeding number shall receive a handsome silver medal on which is the following inscription; 'From the Editors of the *Mathematical Correspondent* to A—B— as a reward of his mathematical merit.'

The small profits arising from the sale of the work will be applied to defraying the expense of the prize medals.

With a view of rendering the work as useful as possible to the generality of our subscribers, we have begun with the lowest parts of mathematics; and for this reason the questions proposed in the first number are such as may be resolved by a very small degree of Mathematical Knowledge. In future, however, we shall gradually ascend towards the higher regions of those sciences, as far as may be thought consistent with the abilities of our readers. On this subject the opinion and advice of every American mathematician will always be thankfully received and duly considered."

The entire content of the *Mathematical Correspondent* is divided into articles, instead of chapters or departments, and numbered from the beginning to the end of the volume.

The first number of Volume I contains four articles, viz., Art. I. "*A New Elucidation of the principles of the Rule of Proportion in Arithmetic; applied to the Resolution of practical questions and to the invention of general rules for making in the neatest and shortest manner possible many of the most useful calculations that daily occur in the counting house.* By G. Baron: Art. II. *A Problem, by William Elliot, printer, New York;* and Art. III. *A Curious and useful proposition in Geometry, with its application to a certain subject in Geography and Navigation.* By R. Tagart.

Mr. Baron, in the course of the discussion of his subject, calls attention to the very obvious fact, well known to every good teacher of arithmetic* but unfortunately unknown to untold thousands of teachers who teach arithmetic at the present time, that one denominate number cannot be multiplied by another denominate number.* He says ". . . . 3*l.* cannot be multiplied by 7*l.*, 7 feet cannot be multiplied by 4 feet, nor 6 bears multiplied by 9 asses. All such questions are evidently unscientific and absurd, and serve only to demonstrate the ignorance and stupidity of their authors." The article, which is concluded in the second number, is a good elementary exposition of proportion.

Mr. Elliot's problem is, Given $a : b = c^2 : d^2$, find x in terms of a and b so that $a+x : b+x = c : d$.

Mr. Tagart's proposition which comprises Art. III is: "The radius of a circle intersects the circumference at right angles." The author uses this theorem to prove that every meridian cuts every parallel of latitude at right angles, the whole discussion having for its

*See Cajori's *The Teaching and History of Mathematics in the United States*, for an interesting instance of the ignorance of teachers on this point.

object the refutation of a certain statement made by John Fraser, a teacher of Newport, R. I. He concludes by saying, "It will be recollected that John Fraser, a teacher of Navigation in Newport, R. I., and another person under the signature of *Loxodromicus*, have in the public newspapers lately made various attempts to persuade the people of their state, 'that a ship steering a due east or west course does not sail on a parallel of latitude, but on a certain spiral line which would ultimately carry her to the equator.' They might just as well have said that a due east and west course will carry a ship to the moon."

Under Art. IV, seven problems are proposed for solution, the last of which is a "prize problem" proposed by George Baron and reads as follows:

> In surveying a field in the form of an oblique angled rectilineal triangle, the first or least side measured 24 chains, the second or next greater side measured 37.44 chains: on the third or greatest side grew two large hemlock trees whose distance asunder was 16.80 chains; a straight line drawn from the obtuse angle to the tree nearest the first side, was perpendicular to the second, and another straight line drawn from the same angle to the other tree, was perpendicular to the first side. Required, the third side, the distance of each tree from the obtuse angle, and the area of the field.

No. 1, Vol. I contains the concluding part of George Baron's article on Proportion, the solutions of the seven problems proposed in No. 1, a problem by James Temple, and eleven problems proposed for solution.

The problem proposed and solved by James Temple is as follows: "To determine in what cases it is possible to cut a given rectangle *ABCD* into two parts, which being joined will be a square." No. III, Vol. 1, contains an article by G. Baron, the solutions of the problems proposed in No. II, and ten problems proposed for solution.

On page 47, of this number, G. Baron, in his solution of problem III, No. 2, calls attention to an error in Shepherd's *Columbian Accountant* and says, "This is but a small specimen of what we could give of the numerary talents of Shepherd." This is the first instance of a number of sarcastic and belittling statements made by the chief editor concerning the writings and opinions of many of his contemporaries.

Mr. Baron's article on page 59 is entitled, *A short Disquisition concerning the word POWER, in Arithmetic and Algebra.*

In the article, he shows that the commonly accepted definition, *viz., The powers of any number, are the successive products, arising from the continued multiplication of that number, into itself*, is erroneous.

In explaining the 0th power, he refers to the 73rd problem in Emerson's *Algebra* and in a foot-note says, "Emerson wrote this problem in the English language, but it has lately been translated into *nonsense*, by Jared Mansfield of Connecticutt and published in a

wonderful work which this translator calls *Mathematical Essays*. The infinitely small *nothings* of Connecticutt are infinitely great absurdities." Yet, be it remembered, that from the perusal of these same Essays, Thomas Jefferson, then President of the United States was induced to bring Jared Mansfield into public life by appointing him surveyor general of the North-West Territory.

Number 4, Vol. I contains the solutions of the problems proposed in No. 3, an article entitled, *An Inquiry Respecting the True Definition of Proportional Numbers*, and eleven problems proposed for solution. The first solution of a problem in this number is by a contributor who signs himself, A. Rabbit, Harlem, near New York. Under an N. B., he says, "This question is taken from Shepard's *Columbian Accountant*, page 48, where the answers are wrong, and evidently deduced from an erroneous principle. In this country, authors of Arithmetics have lately sprung up like a parcel of mushrooms and it would have been for the young and rising generation had the former been as harmless as the latter. These upstart authors have most perniciously corrupted, distorted, and degraded the noble and useful science of numbers, and metamorphosed our sons into mere counting machines, moving according to a heterogeneous collection of unscientific and stupid rules. A good book on Arithmetic is much wanted in America; but as long as the wretched productions of Pike, Walsh, Shepard and Co. are encouraged, we cannot expect a man of talents to enroll his name in our list of our numerical authors."

"Prize Problem," 28, is the following: St. Johns, in Newfoundland, is in latitude 47°32′ N., longitude 52°26′ W.; Cape Finistere, in Spain, is in latitude 42°51′52″ N., longitude 9°17′10″ W.; and Cape Barbos, in Africa, is in latitude 22°15′30″ N., longitude 16°40′ W. Now there is a certain point, (on the same hemisphere of earth) which on the arc of a great circle, is equally distant from each of these places; and it is required to determine this distance, the bearings of the three places from the point, the latitude and longitude of the point and the courses and distances from the same point to each of the same three points."

Thomas Mangham, of Quebec, in concluding his solution of this problem makes this scurrilous remark: "It is, however, well known to mathematicians, that an infinite number of courses and distances may be found which will carry a vessel from any one given place to another in the same latitude and that the greatest of these courses is infinitely near 90°, having its corresponding distance infinitely great. This will necessarily be thought a strange doctrine by the deluded deciples of *Bowditch* and *Blunt* who are taught to believe that 'plain

sailing' considers the earth an extended plane, on which the meridians are parallel to one another!! But to such people, I have only to recommend the candid perusal of a small pamphlet, written by G. Baron, and entitled *Exhibition of the Genuine Principles of Common Navigation with a Complete Refutation of the False and Spurious Principles, Ignorantly Imposed on the Public in the New American Practical Navigator.*" Such is the vindictive judgment of one, Thomas Mangham, of Quebec, passed upon the best work on Navigation, written by the foremost American Mathematician of his day, Nathaniel Bowditch, of Harvard College. For the solution of this problem, let us hope not for his invective, Thomas Mangham was awarded the prize.

The editors also make acknowledgment to the Scientific Society of New York for presenting the prize medal on a former occasion to James Temple.

Number 5, Vol. I, contains solutions of the problems proposed in No. 4, an article entitled, *A Disquisition concerning the motion of a ship which is steered on a given point of the compass*, by Robert Adrain, and twelve new problems proposed for solutions. Problem 42, the third of this list, proposed by T. Bulmer, Sunderland, England, reads as follows:

> Prove that any number divided by 9, and the sum of its digits divided by 9, leave equal remainders; and determine whether this is an absolute property of the number 9, or merely an effect of our present notation.

Professor Adrain solves the problem and shows that the property in question belongs to any system of notations whose radix is r, (r, of course, >1).

In a scholium to this problem the editors say, "The Ingenious proposer, Thomas Bulmer,..., after answering the question nearly in the same manner as above adds, 'What ought we to think of our most eminent English mathematicians of the present day, who have greatly informed us that this property of the digit 9, was unknown to the ancients? The ancients could not possibly be acquainted with an *effect* before its *cause* existed; and hence our profound sages might just as well have told us that old Pythagoras was never on board an American gun-boat.' " Question 45, proposed by Eben E. White, Danbury, Conn., reads as follows: "Given one side of a trapizium and two adjacent angles to find the length of two equal sides, contiguous to the given sides, when the area of the trapizium is a given quantity."

The problem is solved by Professor Adrain. In a scholium to the solution, G. Baron says, "Some *negative mathematicians* will probably say that our learned friend ought here to have remembered that the cosine of an obtuse angle is negative. Pray, scientific gentlemen,

can any of you give a correct definition of a negative quantity*? If such a quantity cannot be defined, it must be either a creature of your own imagination or something which you do not rightly understand; and if you do not know what is meant by a negative quantity, you can have no definite idea of a *negative cosine.* Let us dive into the ocean of negative mysteries, when we have sounded its depths with the line of philosophy.... Remember that a solution and its final equation are two things which differ essentially."

The * refers to a foot-note in which he says, "Jared Mansfield, in his *Mathematical Essays,* has attempted to define a negative quantity. Jared's discourse on this subject reminds me of a Connecticutt carle, who said that the moon was exactly the size of a johnny-cake."

(To be continued in the February issue)

A pure mathematician has told me, what I should certainly have expected, that even purely mathematical theorems are not arrived at in the first instance by a method of demonstration. They arise in the mind as a flash, an idea, an inspiration; and much subsequent labor may be necessary before they are established and rigorously proved. . . . Newton had an extraordinary faculty for guessing correctly, sometimes with no apparent data,—as for instance his intuition that the mean density of the earth was probably between 5 and 6 times that of water; while we now know that it is really about $5\frac{1}{2}$,—and he concludes his "Optics" with a whole string of sagacious queries, every one of which is an untested or incompletely tested hypothesis or speculation.—From Oliver Lodge in his *Reason and Belief*. Moffat, Yard and Company, Publishers, New York.

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

Suggestions for Reducing Mortality in Freshman Mathematics

By W. P. HEINZMAN
State College, N. M.

[We welcome any suggestions that may reduce Freshman Mortality in Mathematics. May it not be, however, that our chief concern at present is to conserve the "upper brackets"—to keep at the highest pitch of achievement those who are capable of great and near great accomplishment? Surely these deserve at least as much of our attention as those who are capable of complete or nearly complete failure. It is not that we have done everything possible to prevent failure. (Just what teachers or administrators mean when they say that they have no failures is a bit hard to understand. Perhaps these educators have stricken the *word* "failure" from their lexicon or changed their grading system to "passed with honor", "passed", "didn't pass; with honor." After all, inability to do something need not be stigmatized by dishonor of any kind. But, when, for example, a man tries to jump a ditch ten feet wide and if his "best jump" is only eight feet, obviously, he will fall in—regardless of whether or not we use the word, "failure", to describe his lowly position. Filling in the ditch, or making it narrower by artificial means, or not permitting the man to jump, present a variety of situations which do not alter the implications of the illustration) We should do all we can to "save souls" at the lower level of ability but we must not let the prospective failures monopolize our time and best efforts.—J. SEIDLIN.]

One in every four freshmen in engineering failed to pass their college mathematics in the years 1923 to 1928. The rate may be higher now. When these failing students repeat the course it has been estimated that six in every ten will fail the second time.

Should these failures be reduced? What can be done to help reduce these failures? One professor said, "We have practically no failures in freshman mathematics. This is due, first, to the fact that mathematics is elective and, second, *to our belief that failures can be avoided by proper teaching.*" This seems to be an extreme view, and

we are not hoping for the elimination of all failures, but for the reduction of failures.

The suggestions here advanced for the reduction of freshman failures were obtained from a good sample of returned questionnaires sent to 950 colleges and universities in all parts of this country.

The remedies that were suggested are grouped into the following divisions: I. Administrative Improvement; II. Teacher Improvement; III. Classroom Procedure; and IV. Miscellaneous:

I. ADMINISTRATIVE IMPROVEMENT:

1. Section the students according to ability.
2. Have a number of coaching sections.
3. Better selection of students.
4. Smaller sections.
5. Entrance examinations.
6. Remove mathematics as a required study.
7. Section according to number of units of high school mathematics.

II. TEACHER IMPROVEMENT:

1. Conferences and individual help.
2. Improve the teaching and encourage the students to do their best.
3. Encourage students to confer with instructor *early* in the course and try to "spot" weak students early that they may be helped.
4. Private tutoring.
5. Students to be notified of their standing at least every six weeks if not more often.
6. Slow up the work.
7. Some method which would promptly catch the absentee.
8. Remove the fear of mathematics.
9. Weekly conferences with all students.

III. CLASSROOM PROCEDURE:

1. Supervised study for laggards.
2. Frequent written tests.
3. Conduct class on semi-laboratory fashion with ample supervision.
4. Extra hour of work every week for those deficient in high school training.
5. Extra two hours of work for those deficient in high school training.

IV. MISCELLANEOUS:

1. Better high school preparation.
2. More study and less "Student Activities".
3. More uniform mathematics requirements for college entrance.
4. Standardizing all grammar grades work so as to give the fundamentals of Arithmetic thoroughly.
5. More pride in passing would largely solve the problem.
6. Six hours per week for a half year instead of three hours per week for a whole year.
7. Four years of high school Mathematics.
8. Collateral reading in History of Mathematics.

THE TRUE REASON?

CAMBRIDGE, MASS., December 6, 1939.

Prof. Joseph Seidlin,
Alfred University, Alfred, N. Y.

DEAR SIR:

In the November issue of the NATIONAL MATHEMATICS MAGAZINE, I read an article by Mr. John Ellis Evans, entitled *Why Logarithms to the Base e Can Justly be Called Natural Logarithms*, in which he says that he had found essentially only three reasons for it: (1) e is a quantity which arises frequently and unavoidably in nature, (2) natural logarithms have the simplest derivatives of all the systems of logarithms, and (3) in the calculation of logarithms to any base, logarithms to the base e are first calculated.

I suggest another if not the *true* reason: Going back to the old definition of logarithms we can show easily why logarithms to the base e are called natural.

Logarithms of the terms of a geometric progression

(1) $$1,\quad 1+a,\quad (1+a)^2, \cdots\cdots, (1+a)^n, \cdots\cdots$$

were defined as the corresponding terms in the arithmetic progression

$$0,\quad a,\quad 2a, \cdots\cdots,\quad na, \cdots\cdots$$

The base is the number such that its logarithm is one. Suppose the base is

$$b=(1+a)^n;$$

then $$na=1 \quad \text{or} \quad n=\frac{1}{a}.$$

Therefore $$b=(1+a)^{1/a}$$

In order that every number have a logarithm; that is, in order that every number shall be included in the progression (1), it is necessary that

$$a \to 0. \qquad \text{Then} \quad b \to e.$$

Yours very sincerely,
MARIO O. GONZALES.

Mathematical World News

Edited by
L. J. Adams

In volume 90, number 2344 of *Science* Professor G. A. Miller (University of Illinois) published a list of a dozen errors in the Encyclopedia Brittanica. The errors occurred in articles relating to mathematics.

Professor Joseph Seidlin has been appointed Director of the Graduate Division, College of Liberal Arts, Alfred University.

On November 24, 1939 Professor Joseph Seidlin addressed the N. I. T. P. A. at their annual meeting in Chicago on *General Educational Values Inherent in the Professional Program for Teachers.* On December 28, 1939 Professor Seidlin was scheduled to speak before Section Q of the A. A. A. S. on *The Effect of Hiring Agencies on Teacher Selection and Training.*

On December 9, 1939 the Tri-County Mathematics Association met at Alfred University. Mr. H. C. Taylor, of the Rochester School system, spoke on the subject *You'll Have to Transfer.*

Oliver and Boyd, publishers at Tweeddale Court, Edinburgh, announces the following books published recently in a uniform edition:

1. *Determinants and Matrices.* A. C. Aitken.
2. *Integration.* R. P. Gillespie.
3. *Vector Methods.* D. E. Rutherford.
4. *Integration of Ordinary Differential Equations.* E. L. Ince.
5. *Statistical Mathematics.* A. C. Aitken.
6. *Theory of Equations.* H. W. Turnbull.

The price is $1.25 per volume, plus postage.

G. Bell and Sons (London) announces the publication of the *James Gregory Tercentenary Memorial Volume* for the Royal Society of Edinburgh. This volume contains the correspondence of James Gregory with John Collins, Gregory's hitherto unpublished mathematical manuscripts and his addresses and essays communicated to the Royal Society of Edinburgh. The volume is edited by Herbert Westren Turnbull, Regius Professor of Mathematics in the United College, University of St. Andrews.

The Louisiana-Mississippi Section of the M. A. A. and the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics meets at the University of Mississippi, March 8-9, 1940. Professor W. P. Carver of Cornell University, President of the M. A. A. is to be guest speaker at the meeting.

Problem Department

Edited by
Robert C. Yates and Emory P. Starke

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to Robert C. Yates, Mathematics, University, Louisiana.

SOLUTIONS

No. 284. Proposed by *J. Rosenbaum*, Bloomfield, Connecticut.

Find an analogous formula for a tetrahedron that corresponds to the Law of Cosines for a triangle.

Solution by *Johannes Mahrenholz*, Cottbus, Germany.

Let F_1, F_2, F_3, F_4 represent the face areas of the tetrahedron and (i,k) the angle between the planes of F_i and F_k. Then

$$F_1 = F_2 \cos(1,2) + F_3 \cos(1,3) + F_4 \cos(1,4)$$

$$F_2 = F_3 \cos(2,3) + F_4 \cos(2,4) + F_1 \cos(2,1)$$

$$F_3 = F_4 \cos(3,4) + F_1 \cos(3,1) + F_2 \cos(3,2)$$

$$F_4 = F_1 \cos(4,1) + F_2 \cos(4,2) + F_3 \cos(4,3).$$

Accordingly,

$$F_2^2 + F_3^2 + F_4^2 - F_1^2 = 2F_2F_3 \cos(2,3) + 2F_3F_4 \cos(3,4) + 2F_4F_2 \cos(4,2),$$

or

$$F_1^2 = F_2^2 + F_3^2 + F_4^2 - 2F_2F_3 \cos(2,3) - 2F_3F_4 \cos(3,4) - 2F_4F_2 \cos(4,2).$$

Also solved by *C. W. Trigg* who gives the expression:

$$F_4^2 = F_1^2 + F_2^2 + F_3^2 - \frac{a_1a_2a_3}{2}\,[a_1(\cos\phi_1 - \cos\phi_2\cos\phi_3)$$
$$+ a_2(\cos\phi_2 - \cos\phi_1\cos\phi_3) + a_3(\cos\phi_3 - \cos\phi_1\cos\phi_2)],$$

where a_1, a_2, a_3 are edges of the tetrahedron meeting in one vertex with opposite angles ϕ_1, ϕ_2, ϕ_3 contained in the faces F_1, F_2, and F_3.

No. 293. Proposed by *Howard D. Grossman*, New York City.

Nim is a game described in books on mathematical recreations. Small similar objects are divided into any number of groups with any number in each group. Two players play alternately, each removing from any one group any positive number of objects. He who removes the last of all the objects wins the game.

A solution is known to be as follows: Express at each move the number of objects in each group in the scale of 2 and arrange these numbers one under another; then play so as to leave the sum of the digits in each column an even number. If a player once meets this condition, his opponent must destroy it and he can restore it. Prove this solution.

Prove also this generalization: If the game is played so that any positive number of objects may be removed from any number of groups $\leq k$, then the solution is, expressing and arranging the numbers as before, to leave the sum of digits in each column $\equiv 0 \bmod (k+1)$. If a player once meets this condition, his opponent must destroy it and he can restore it.

Note by the *Editors*.

Discussions of Nim and proofs of the proposed solution may be found in Hardy and Wright, *Introduction to the Theory of Numbers* (Oxford, 1938), pp. 116-119, and in Uspensky and Heaslet, *Elementary Number Theory*, (New York, 1939), pp. 16-19.

No. 294. Proposed by *Walter B. Clarke*, San Jose, California.

Construct a square whose sides, prolonged if necessary, will each pass through one of four arbitrary points in a plane. How many such squares exist?

Solution by *W. T. Short*, Oklahoma Baptist University, Shawnee.

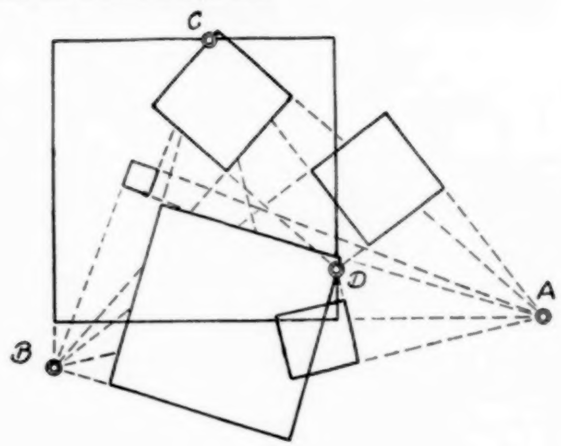
Let P_1, P_2, P_3, P_4 be the four arbitrary points. Let K be the midpoint of the semi-circle described upon P_1P_2 as a diameter; L be the midpoint of the semi-circular arc upon P_3P_4 as diameter. Draw KL cutting the first circle at A and the second circle at C. Draw AP_2 and CP_3 meeting at B. Draw AP_1 and CP_4 meeting at D. Then $ABCD$ is the required square.

Proof: The triangles ABC and ADC are each isosceles right triangles since each of the base angles of these triangles intercepts an

arc of 90 degrees. Therefore, angle B = angle D = 90°. In general there are *six* solutions. After P_1 is selected the other three points can be designated as P_2, P_3, P_4 in six different ways.

Indeterminate solutions arise in case K and L coincide. The line determining A and C then could be any line through this common point. In that case, however, the line joining P_1 and P_3 and that joining P_2 and P_4 would have the same length and be perpendicular each to each.

Also solved by *D. L. MacKay, Johannes Mahrenholz, C. W. Trigg,* and the *Proposer* who submits the following figure of the general case displaying the six distinct squares.



References:

Morley and Morley, *Inversive Geometry*, p. 210.

L'Education Mathématique, 1927, No. 5, Question 5273. (Only four solutions given here). (MacKay).

Biandsutter, *Mathesis*, 1881, p. 8 (MacKay).

Altshiller-Court, *College Geometry*, p. 19, Prob. 38. (Trigg).

E. de la Garza, *School Science and Mathematics*, January, 1929, p. 94, Prob. 1034. (Trigg).

American Mathematical Monthly, October, 1928, p. 400. (Trigg).

School Science and Mathematics, February, 1938, p. 222, Prob. 1524. (Trigg).

No. 296. Proposed by *W. V. Parker*, Louisiana State University.

Prove that the ellipse of least area circumscribing any right triangle has eccentricity not less than $\sqrt{(2/3)}$.

Solution by *C. W. Trigg*, Los Angeles City College.

The ellipse of least area circumscribing a triangle is the ellipse in No. 257 (2), page 382, May, 1939. Consider a right triangle with vertices (0,0), $(a,0)$ and $(0,b)$. Then the equation of the required ellipse is

$$\begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ a & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} x & y & 1 \\ 0 & b & 1 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ a & 0 & 1 \\ 0 & b & 1 \end{vmatrix} \cdot \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ a & 0 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ 0 & b & 1 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} x & y & 1 \\ a & 0 & 1 \\ 0 & b & 1 \end{vmatrix} = 0.$$

This reduces to $\quad b^2x^2+abxy+a^2y^2-ab^2x-a^2by=0.$

The eccentricity of the ellipse is the positive real root of*

$$(b^2a^2-\tfrac{1}{4}a^2b^2)e^4+[(b^2-a^2)^2+a^2b^2]\,(e^2-1)=0.$$

That is,

$$e=\sqrt{\frac{2}{3}}\sqrt{\frac{(a^2+b^2)\sqrt{b^4-a^2b^2+a^4}-(b^4-a^2b^2+a^4)}{a^2b^2}}$$

$$=\sqrt{\frac{2}{3}}\sqrt{(k+1/k)\sqrt{1/k^2-1+k^2}-(1/k^2-1+k^2)},$$

where $k=a/b$. The value of the expression under the second radical is clearly a minimum, and equal to unity, when $k=1$. Hence the minimum value of e is $\sqrt{2/3}$, for which the right triangle is isosceles.

No. 297. Proposed by *H. T. R. Aude*, Colgate University.

Each of the two equations

$$30x+23y=c,$$

$$30x+23y=c+1$$

has only one solution in positive integers. Show that the greatest value of c is more than three times the least value of c.

Solution by *C. W. Trigg*, Los Angeles City College.

If each of the equations $30x+23y=c$ and $30w+23z=c+1$ has a single solution† in positive integers for a particular value of c, then neither x nor w may exceed 23, nor may y nor z exceed 30. Furthermore $30(w-x)+23(z-y)=1$ has solutions, which are quickly found from $(w-x)=[1-23(z-y)]/30$. Only two of these solutions meet the

*C. Smith, *Conic Sections*, (*1884*), page 206.

†For a general treatment of the number of sets of solutions in positve integers for such a Diophantine equation, see Hall and Knight, *Higher Algebra*, pp. 287-289, 291. —ED.

restrictions on x, y, z and w. From the first set we find the minimum and maximum values of c to be 352 and 1080 from

$$(x, y, z, w) = (1, 14, 1, 11) \quad \text{and} \quad (13, 30, 17, 23).$$

Likewise the second set yields $c = 443$ and 989. Now $1080 > 3 \cdot 352 = 1052$, as required.

No. 298. Proposed by *Howard D. Grossman*, New York City.

Let a, b, c, be the sides of any triangle, G its centroid, I the incenter, and P the centroid of its perimeter. If parallels to the sides be drawn through G, any side b is divided *internally* in the ratio 1:1:1; if through I, in the ratio $a{:}b{:}c$; if through P, in the ratio

$$(b+c) : (c+a) : (a+b).$$

Solution by *Walter B. Clarke*, San Jose, California.

Take an arbitrary point O internal to the triangle and draw the lines determined by this point and the vertices. Let these lines (the cevians) meet the sides a, b, c in the points L, M, N. Let the parallels to the sides a and c meet b in the points D and E, respectively. Then by similar triangles:

$$(CD)/b = (OL)/(AL); \quad (EA)/b = (ON)/(CN).$$

Thus
$$\begin{aligned} DE &= b - b(OL)/(AL) - b(ON)/(CN) \\ &= b[1 - (OL)/(AL) - (ON)/(CN)]. \end{aligned}$$

But $(DE)/b = (OM)/(BM)$. Therefore

$$R \equiv (CD) : (DE) : (EA) = (OL)/(AL) : (OM)/(BM) : (ON)/(CN).$$

Thus it is only necessary to know how any point divides its cevians in order to give the appropriate division of sides by lines through the point parallel to the sides.

(1) For $O = G$, the centroid, the preceding ratios are each $\frac{1}{3}$, and thus $R = 1{:}1{:}1$.

(2) For $O = I$, the incenter, the ratios are $a/(2s)$, $b/(2s)$, $c/(2s)$ where $s = (a+b+c)/2$. Thus $R = a{:}b{:}c$.

(3) P is the incenter of the medial triangle. (See No. 253, p. 290, this Magazine, March, 1939). The foregoing ratios are $b(b+c)/4s$; $b(c+a)/4s$; $b(a+b)/4s$ and $R = (b+c) : (c+a) : (a+b)$.

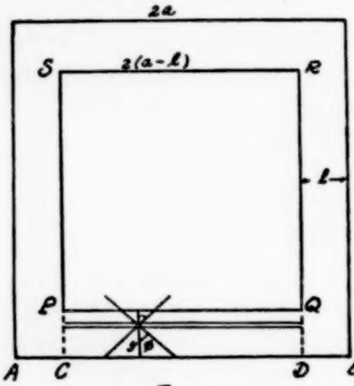
(4) If O coincides with the verbicenter, the ratios are $b(s-a)/s$; $b(s-b)/s$; $b(s-c)/s$ and $R = (s-a) : (s-b) : (s-c)$.

Also solved by *D. L. MacKay*, *L. M. Kelly*, *Johannes Mahrenholz*, *C. W. Trigg*, and the *Proposer*.

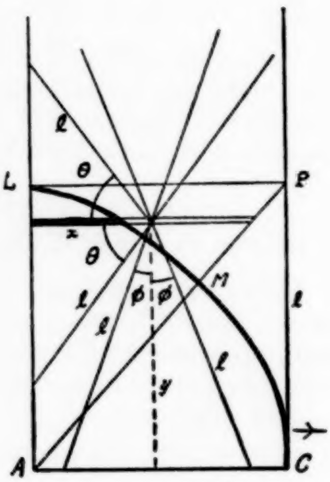
No. 299. Proposed by *R. E. Gaines*, University of Richmond.

A slender rod of length $2l$ rests on a square table the length of whose side is $2a(l<a)$. Find the probability (1) that both ends of the rod are on the table and (2) that both ends of the rod extend over the edge of the table.

Solution by the *Proposer*.

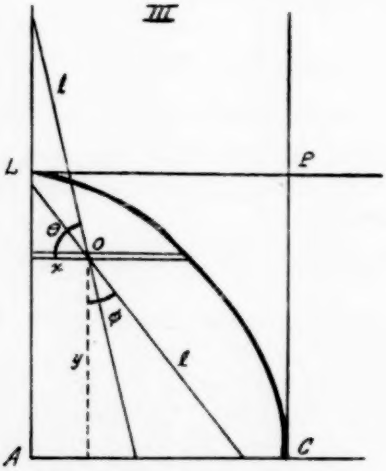


I



II

III



(i). In the center of the table draw a square $PQRS$ such that each side is parallel to a side of the table and has the length $2(a-l)$. If the center, O of the rod falls within this square both ends will lie on the table. The probability for this is $4(a-l)^2/4a^2$ or $(a-l)^2/a^2$.

(ii). If O falls in one of the four rectangles $CDQP$, (see Fig. I), the direction of the rod must be taken into account. The probability that O will fall in the narrow band of width dy is $2(a-l)dy/4a^2$; and the probability that then both ends of the rod will lie on the table is $(\pi-2\varphi)/\pi$, where $\cos\varphi=y/l$. The product of these multiplied by 4 (since there are four such rectangles) and integrated with respect to y is

$$\frac{2(a-l)}{\pi a^2}\int_0^l(\pi-2\varphi)dy=\frac{2(a-l)}{\pi a^2}\Big[\pi y-2\varphi y+2l\sin\varphi\Big]_{y=0,\ \varphi=\pi/2}^{y=l,\ \varphi=0}$$

$$=2(a-l)l(1-2/\pi)/a^2.$$

(iii). If O falls in the area bounded by the lines LP and PM and the circular arc LM* (see Fig. II), we have

$$\frac{\pi-2\varphi-2\Theta}{\pi}\quad\frac{dxdy}{4a^2}$$

to integrate, multiplied by 8 since there are eight such regions. Hence the probability is

$$\frac{2}{\pi a^2}\int_{l/\sqrt{2}}^{l}\int_{\sqrt{l^2-y^2}}^{y}(\pi-2\varphi-2\Theta)dxdy=\frac{l^2}{a^2}\left(1-\frac{3}{\pi}\right).$$

The solution for problem (1) is therefore the sum of (i), (ii), and (iii), and we have $1-l(4a-l)/\pi a^2$.

For case (2) in which both ends of the rod project over the edge of the table (see Fig. III) we use similar argument to obtain the probability

$$\frac{1}{2\pi a^2}\int_0^l\int_0^{\sqrt{l^2-y^2}}(2\varphi+2\Theta-\pi)dxdy$$

$$=\frac{1}{2\pi a^2}\int_0^l 2l(1-\cos\varphi)dy=\frac{l^2}{2\pi a^2}.$$

No. 301. Proposed by *V. Thébault*, Le Mans, France.

Determine the system of numeration in which a number of the form *abcabc* is the cube of a number *dd*.

Solution by *Johannes Mahrenholz*, Cottbus, Germany.

*The circular arcs are the loci of the centers of the rod as it takes limiting positions, that is, with both ends on the edges of the table.—ED.

With B for the base of the system, the hypothesis may be written

$$(B^3+1)(aB^2+bB+c)=d^3(B+1)^3$$

or

$$(B^2-B+1)(aB^2+bB+c)=d^3(B+1)^2.$$

If B and d are so chosen that $B^2-B+1=d^3$, then a, b and c can be selected such that $aB^2+bB+c=(B+1)^2$. Thus with $B=19$, $77^3=121121$ is a solution. Then evidently also $\overline{14}\ \overline{14}^3=8\ \overline{16}\ 8\ 8\ \overline{16}\ 8$.

Also solved by the *Proposer*.

EDITOR'S NOTE: Since $(B+1)^2-(B^2-B+1)=3B$ while B is relatively prime to $(B+1)^2$, the highest common factor of $(B+1)^2$ and (B^2-B+1) is 1 or 3. Then, of necessity, we have either

(1) $\qquad aB^2+bB+c=k(B+1)^2$ and $B^2-B+1=d^3/k$ or

(2) $\qquad aB^2+bB+c=k(B+1)^2/3$ and $B^2-B+1=3d^3/k$,

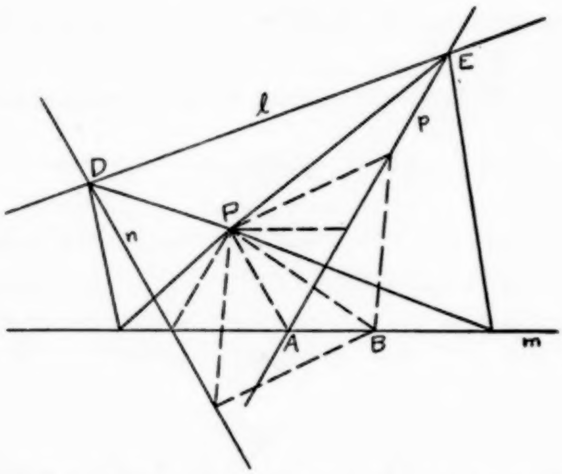
with $k<B+1$ or $k<3(B+1)$ in the respective cases. The above solutions come under (1) with $k=1$ or 8. Further solutions under (1) are hard to find, but with the substitution $d=k$ (2) yields infinitely many solutions. We have $B^2-B+1=3d^2$ which reduces to $3u^2+1=v^2$ upon putting $2B-1=3u$ and $2d=v$. Solutions, with v even are given by $v_0=2$, $u_0=1$, $v_{n+1}=7v_n+12u_n$, $u_{n+1}=4v_n+7u_n$. Thus the first three solutions are $B=2$, $11^3=011011$; $B=23$, $\overline{13}\ \overline{13}^3=4\ \overline{16}\ \overline{12}\ 4\ \overline{16}\ \overline{12}$; $B=314$, $\overline{181}\ \overline{181}^3=\overline{60}\ \overline{225}\ \overline{165}\ \overline{60}\ \overline{225}\ \overline{165}$.

No. 302. Proposed by *Walter B. Clarke*, San Jose, California.

Construct an equilateral triangle whose vertices lie at a given point and upon each of two given lines.

Solution by *C. W. Trigg*, Los Angeles City College.

It has been shown (No. 252, page 289, March, 1939) that the locus of the third vertices of the similar triangles, whose first vertices are at a fixed point and whose second vertices lie on a fixed line, is a pair of lines. Hence, in order to construct equilateral triangles with vertices at P and on lines l and m, with P as center and arbitrary radii strike arcs meeting m at A and B. Construct equilateral triangles on both sides of PA and PB. The third vertices of these triangles determine the two lines n and p which intersect l at D and E. On PD and PE construct equilateral triangles whose third vertices will fall on m, since they are members of the families of triangles which determined the locus.



If l is parallel to either p or n, there will be but one solution.

A similar method could be employed to construct triangles *similar to a given triangle* whose vertices lie at a given point and upon each of two given lines. In this case there would be, generally, six solutions.

Another method, which involves drawing a line through P which will pass through the intersection of l and m, is given by Kate Bell, *School Science and Mathematics*, problem 1038, page 198, February, 1929.

Also solved by *D. L. MacKay*, *Johannes Mahrenholz*, and the *Proposer*.

No. 303. Proposed by *H. T. R. Aude*, Colgate University.

Find the smallest quadrilateral with unequal integral sides such that each of two opposite angles is 120°.

Solution by *C. W. Trigg*, Los Angeles City College.

Let the sides including the 120° angles be x, y and u, v, and let the diagonal connecting the vertices of the other two angles be d. Then by the law of cosines,

$$d^2 = x^2 + y^2 - 2xy \cos 120° = x^2 + xy + y^2$$

and

$$d^2 = u^2 + uv + v^2.$$

It is required to find the least value of d^2 such that $d^2 = x^2 + xy + y^2$ has two solutions. The theory of this equation is given in the solution of Problem No. 268 (this Magazine, Vol. XIII, No. 8, May, 1939, p. 388) where the smallest solution is shown to be $x=1$, $y=9$, $u=5$, $v=6$,

$d^2=91$. Thus the required quadrilateral has sides 1, 9, 5 and 6 and area $39\sqrt{3}/4$.

Also solved by *Annie Christensen*, *W. B. Clarke*, and the *Proposer*.

PROPOSALS

No. 332. Proposed by *C. W. Trigg*, Los Angeles City College.

On side AB of the triangle ABC as base are constructed two isosceles triangles with 120° vertex angles, C' exterior to ABC and C'' interior to ABC. The same procedure is followed on the other two sides of ABC.

(1) $A'B'C'$ and $A''B''C''$ are equilateral.

(2) ABC, $A'B'C'$, and $A''B''C''$ have the same centroid.

(3) Find the sides of the equilateral triangles in terms of the sides of the given triangles.

(4) Under what circumstances will the vertices of one of the equilateral triangles fall upon the sides of the other?

No. 333. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

Prove that the locus of the feet of the perpendiculars drawn from the vertex of a triangle upon the polars of that vertex with respect to the circles of the coaxal pencil determined by the other two vertices is an Appolonian circle.

No. 334. Proposed by *H. L. Smith*, Louisiana State University.

Consider the set of all points (x,y) which satisfy the inequalities

$$x^2+y^2 \leqq r^2, \quad r>0$$

$$|x+y|+|x-y| \leqq 2.$$

Let $A(r)$ denote the area of that set. Determine the minimum value of $r/A(r)$.

No. 335. Proposed by *Nathan Altshiller-Court*, University of Oklahoma.

Find the locus of the point common to the polar planes, with respect to three given spheres, of a variable point describing a plane perpendicular to the plane passing through the centers of the given spheres.

No. 336. Proposed by *V. Thébault*, Le Mans, France.

Find three digits, a, b, c, such that each of the numbers $a00b0c$ and $a00b00c$ is a perfect square.

No. 337. Proposed by *W. V. Parker*, Louisiana State University.

Given the triangle $A_1A_2A_3$. Perpendiculars to A_1A_2 at A_1 and A_2A_3 at A_2 meet in P. Perpendiculars to A_1A_2 at A_2 and A_2A_3 at A_3 meet in Q. Prove that the line PQ passes through the circumcenter of $A_1A_2A_3$.

No. 338. Proposed by *Daniel Arany*, Budapest, Hungary.

Put $$a \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{and} \quad b \equiv \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix},$$

further
$$d \equiv \begin{vmatrix} a_{12}b_{13}-a_{13}b_{12} & a_{22}b_{23}-a_{23}b_{22} & a_{32}b_{33}-a_{33}b_{32} \\ a_{13}b_{11}-a_{11}b_{13} & a_{23}b_{21}-a_{21}b_{23} & a_{33}b_{31}-a_{31}b_{33} \\ a_{11}b_{12}-a_{12}b_{11} & a_{21}b_{22}-a_{22}b_{21} & a_{31}b_{32}-a_{32}b_{31} \end{vmatrix}$$

and
$$D \equiv \begin{vmatrix} A_{12}B_{13}-A_{13}B_{12} & A_{22}B_{23}-A_{23}B_{22} & A_{32}B_{33}-A_{33}B_{32} \\ A_{13}B_{11}-A_{11}B_{13} & A_{23}B_{21}-A_{21}B_{23} & A_{33}B_{31}-A_{31}B_{33} \\ A_{11}B_{12}-A_{12}B_{11} & A_{21}B_{22}-A_{22}B_{21} & A_{31}B_{32}-A_{32}B_{31} \end{vmatrix}$$

where A_{ik} and B_{ik} are the cofactors of a_{ik} and b_{ik} respectively. If $a \neq 0$, $b \neq 0$, show that $D = abd$.

NOTE: In No. 323 (this Magazine for November) the summations in the left members are not defined for $r=0$. They should be printed as follows:

(1) $$2^{2n-1}\left[1+\sum_{r=1}^{n-1}\frac{(2n-r-1)\cdots(2n-2r)}{r!}\right].$$

(2) $$2^{2n}\left[1+\sum_{r=1}^{n}\frac{(2n-r)\cdots(2n-2r+1)}{r!}\right].$$

Bibliography and Reviews

Edited by
H. A. SIMMONS

Formal Logic: a modern introduction. By Albert A. Bennett and Charles A. Baylis. Prentice-Hall, Inc., New York, 1939, viii+407 pp.

This book is intended to serve as a text-book for a first course in logic. No previous familiarity with logic is assumed, and only slight knowledge of mathematics. Each chapter is provided with exercises and references for further study.

I here give a brief, and hence not perfectly accurate, account of the symbolism used in this review. "x" stands for an arbitrary individual, "A" for an arbitrary class, "1" for the universal class, "Φ" for an arbitrary propositional function, and "p" and "q" for arbitrary propositions. The letter "ϵ" is used to express membership of an individual in a class, so that "$x\epsilon A$" means that the individual x belongs to the class A; in the usual way, I write "$\Phi(x)$" to mean that the individual x has the property Φ. The expression "(x)" standing before a proposition, expresses the condition that this proposition is true for every value of x; thus "$(x)\Phi(x)$" means that every individual has the property Φ. The symbol "$\sim$" before a proposition expresses the fact that the proposition is false; thus "$\sim(x\epsilon A)$" means that x is not a member of the class A. The symbol "$\Diamond$" before a proposition expresses the fact that it is possible that this proposition is true; thus "$\Diamond(x)\Phi(x)$" means that no contradiction is involved in supposing that everything has the property Φ. The symbol "$\subseteq$" occurring between two propositions means that the second is a logical consequence of the first (this corresponds to Lewis' "strict implication"). The symbol "$\dot{\subseteq}$" occurring between two propositions means that if the first is true, then the second also, as a matter of fact, is likewise true. (This corresponds to Russell's "material implication.") The symbol "C" standing before two propositions expresses the fact that the first implies the second in the sense of Lukasiewicz and Tarski.

Of the thirteen chapters, the first eleven are concerned with deductive logic. The treatment of the topics here has been strongly influenced by the ideas of C. I. Lewis regarding strict implication. The notion of "strict" relations in logic is extended in such a way, that one can speak of "strict inclusion" of one class in another, and the like. (To say "A is strictly included in B", for example, means "For every

x, it is impossible that x belong to A and not to B.") Allowance is also made, of course, for "material" logical relations. Thus the book provides for a rather complicated apparatus of the strict and the material, which an elementary student might find difficulty in keeping straight. Indeed, it seems to the reviewer that the authors are not themselves always clear regarding which symbols are to stand for strict relations and which for material relations. This is particularly the case with respect to expressions of the form "$\Pi_{x\epsilon 1}\Phi(x)$" which are apparently sometimes used to mean what would be ordinarily written "$(x)\Phi(x)$" and sometimes for what might be written "$(x)\sim\Diamond\sim\Phi(x)$". This confusion is made most apparent when we consider the following statements on page 293: "$\Pi_{x\epsilon A}\cdot\Phi(x)$ is equivalent to $\Pi_{x\epsilon 1}: x\epsilon A \dot{\subseteq}\cdot\Phi(x)$" and "$\dot{\Pi}_{x\epsilon A}\cdot\Phi(x)$ is equivalent to $\Pi_{x\epsilon 1}: x\epsilon A\subseteq\Phi(x)$". (The symbol "$\dot{\Pi}$" is intended to be the material sign analogous to the strict sign "Π".) If we take these statements literally, we should conclude from the first (by replacing "A" by "1" and "$\Phi(x)$" by "$x\epsilon A\cdot\dot{\subseteq}\Phi(x)$" that $\Pi_{x\epsilon A}[x\epsilon A\dot{\subseteq}\Phi(x)]$ is equivalent to $\Pi_{x\epsilon 1}\{x\epsilon 1\subseteq\cdot[x\epsilon A\dot{\subseteq}\Phi(x)]\}$. Hence, by the second statement, $\dot{\Pi}_{x\epsilon A}\cdot\Phi(x)$ is equivalent to

$$\Pi_{x\epsilon 1}\{x\epsilon 1\cdot\subseteq[x\epsilon A\dot{\subseteq}\Phi(x)]\}.$$

But it is doubtful, to say the least, whether the authors would be willing to accept this last equivalence, since it expresses the presumably material symbol $\dot{\Pi}_{x\epsilon A}\cdot\Phi(x)$ in terms of a formula which seems to depend in an essential way on the strict symbol "$\subseteq$".

More generally, it should be pointed out that the authors do not discuss, or even mention, the suggestion of Rudolf Carnap that the modal notions "(strict implication", "possibility", etc.) can conveniently be regarded as snytactical in nature. The reviewer is of the opinion that so to regard the modal notions would greatly simplify and clarify discussion of them.

On pages 278-280 will be found some discussion of many-valued logics. It will be noticed that the three-valued matrix given here for "Cpq" differs from the one given by Lukasiewicz and Tarski—in that $CMM=M$ in this matrix, whereas $CMM=T$ in the Lukasiewicz-Tarski matrix. Thus, while the formula Cpp assumes the value T in the Lukasiewicz-Tarski matrix, such is not the case with the matrix here given. The authors state that an interpretation can be obtained for their matrix by supposing that "T" means "known to be true", that "F" means "known to be false", and that "M" means "not known to be true and not known to be false". It is not stated how "C" is to be read in this interpretation, but it is presumably to be read "if..., then...". The reviewer is of the opinion, however, that

if this is what is intended, then the proposed interpretation is defective. For "Fermat's last theorem is true" is not known to be true and not known to be false; but "if Fermat's last theorem is true, then Fermat's last theorem is true" is known to be true.

In Chapter 12 the authors discuss probability; they are inclined here to follow the views of J. M. Keynes, but some space is allowed for a discussion of the frequency theory. The reviewer believes that the objections here raised to the frequency theory have since been answered by Ernest Nagel in his recent *Principles of the theory of probability*, International Encyclopædia of Unified Science, Vol. 1, No. 6.

The last chapter constitutes an introduction to the theory of induction and scientific method.

Viewing the book as a whole, we feel that a formal deductive presentation of logic is unhappily, and rather unnecessarily, avoided. Elementary philosophy students might well be expected to grasp a deductive development of at least the calculus of propositions; and it is doubtful whether any very solid understanding of modern logic can be acquired except by the hard way of mastering the technique of formal proof.

New York University. J. C. C. MCKINSEY.

Differential and Integral Calculus (Second Edition). By John Haven Neelley and Joshua Irving Tracey. The Macmillan Company, New York, 1939, ix+495 pages.

The authors state in the preface that this text is presented with the belief that it is well adapted for use both in Academic colleges and in engineering schools. Their aim therefore was two-fold: to make clear the underlying principles of the subject and to equip the student with a technique for applying the calculus to the various fields of science and engineering. The book meets these objectives with a high degree of success, and in the reviewer's opinion the book is as well suited for use in American colleges and engineering schools as most of the more popular elementary texts on the subject.

Besides the usual material found in a first course in the calculus, this book contains three chapters on analytic geometry and three on differential equations. The part that deals with the calculus and differential equations can be covered in eight semester hours.

To indicate the difference between the new and the old edition, we quote from the preface to the second edition: "To increase classroom utility many additional carefully chosen problems have been

interspersed throughout the text; especially those of a less involved nature, for purposes of drill, and on the other hand, a number of highly challenging problems."

In regard to rigor of proofs the authors have accomplished (with few exceptions) what they set out to do according to the preface, that is "to use only proofs which are valid but which may involve certain assumptions, the proof of which belongs properly in an advanced course." The assumptions are usually explicitly stated.

Some of the other commendable features of the book are the clear style, the large selection and careful arrangement of the exercises and practical problems, and the attractive appearance of the book.

The following is a list of the few places which were noticed by the reviewer to require corrections or changes:

Page 73. The examples of limits do not conform with the definition of the limit given in the text.

Page 83. The definition of the derivative of a function as stated in the text implies that a continuous function must possess a derivative.

Page 95, line 3. It should be specified that $\Delta y \neq 0$.

Page 193, footnote. In the statement of Rolle's theorem it is sufficient that $F'(x)$ exist not everywhere but only at each interior point of the interval in question. $F'(x)$ is ordinarily not defined at the endpoints of the interval.

Page 299. In the statement "...such infinitesimals may be discarded, since the limit of their sum as the number increases indefinitely is zero in all cases with which we shall deal" the last eight words encourage the student to be careless in working problems of this text, and handicap him in solving problems arising in other connections. We would prefer to replace "since" by "if" and omit the last eight words of that statement.

Page 430, first paragraph. A definite conclusion is drawn from a premise which admittedly only *seems* to be true. It would be better to omit such an attempt at a proof.

Integration and the definite integral are defined in terms of the inverse of differentiation.

College of St. Thomas. H. P. THIELMAN.

Freshman Mathematics. By G. W. Mullins and D. E. Smith, Boston, Ginn and Company, 1927. vi+386 pages.

For several years this book has been successful in fulfilling the purpose for which it is intended. The aim of the authors is to present to students who do not plan to continue the study of mathematics

beyond the calculus a fairly comprehensive survey of the mathematics usually included in first year college work, with the emphasis placed upon its usefulness in the various fields of intellectual activity. The book is an introduction to those topics from college algebra, trigonometry, analytical geometry, and calculus which are most likely to be of some practical use. The material presented is intended for a one year course, at the end of which the student should have a sufficient working knowledge of the mathematics required in first courses of the various sciences. Only an elementary knowledge of algebra, plane geometry, and mensuration is assumed as a prerequisite.

At the outset, the reader is introduced to the notion of the "formula" and of its obvious need in engineering and other practical problems. Factoring, linear, simultaneous and quadratic equations, graphs, frequency curves, the notion of function, and exponents complete the opening chapter on algebra. The practical need for each new idea as it is developed is constantly brought to the reader's attention. The next two chapters on algebra continue with the binomial theorem, including the notion of mathematical induction, progressions, and logarithms.

After devoting eighty-two pages to college algebra, the authors turn to trigonometry, which is admirably dealt with, as far as it goes, in one long chapter of another eighty pages. The trigonometrical ratios are introduced as functions of the general angle. The usual treatment follows, leading up to solutions of triangles with applications also to problems involving the law of the parallelogram of velocities. Four-place tables are given at the end of the text. The extensive field of analytical trigonometry is dealt with briefly in two pages at the end of the chapter, and the study of the addition formulas, multiple and half-anlge formulas and inverse functions is virtually omitted.

Chapter V, on analytical geometry, includes the conic sections and such curves as the cissoid, conchoid, exponential and trigonometrical loci. The authors omit, among other things, transformations of coordinates and polar coordinates.

The calculus in the following seventy-eight pages of the text is introduced by presenting a practical problem, involving the notion of the maximum of a function. The rules that follow are applied only to algebraic functions. Applications are made to rates and maximum and minimum problems. The concept of increment had been introduced earlier in the chapter on geometry. Integration is brought in as the process inverse to differentiation. The trapezoidal rule for areas, work problems, length of curve, momentum and other concepts from physics are introduced, the emphasis again being placed on the practical application of such concepts.

With a knowledge of the derivative, the student is now in a position to turn to the study of numerical equations from the point of view of Newton's tangent method in a short chapter which follows. The book closes with a chapter containing a brief study of mensuration of geometric plane and solid figures, in order to give training in spatial perception.

The problems are well chosen and of a practical nature, although some readers may regret that no answers are given. The explanations throughout the book are clear. A noteworthy feature is the frequent presentation of historical notes which stimulate the interest of the reader.

Rutgers University. M. S. ROBERTSON.

Research and Statistical Methodology, Books and Reviews, 1933-1938. By Oscar Krisen Buros, Rutgers Press, New Brunswick, N. J., 1938. vi+99 pages.

A new idea in books. The book department of any professional magazine is considered by both editors and readers to be an indispensable part of every issue. Editors select reviewers with care with the hope that the reviews which are written will constitute significant appraisals of what is being issued by the press in any given field. The need for such appraisals is unquestionably great, considered, in fact, to be of such importance that the editors of *The Annals* of the American Academy of Political and Social Science, for instance, devote some 75 or 80 pages of each issue to thought provoking discussions of new books. Every magazine has its book department with refreshing reviews. But even so, if one wishes to find a good review of some recent book, how may he know where such a review may be found? Or, if he wishes to compare the printed appraisals of all the books issued in a given field, just how will he find time or energy, assuming he possesses the skill, to locate all these reviews?

A recent book remedies this difficulty, at least for a time, in the field of educational statistical methodology. The editor has located the critical reviews of statistical methodology books, has excerpted from these the more significantly evaluative statements, and has brought them together as a single bound treatise. If, then, one wishes to know what critical reviewers have thought of the books issued in this field between the years of 1933-38, all one needs to do is to consult this book.

Professor Buros' idea is a capital one, and he has carried out his idea with great care and skill. His volume affords a well conceived

and satisfactorily executed survey of existing books on statistical methodology. For the specific purpose for which it was written it would be difficult to conceive how it could have been better done.

Northwestern University. J. M. HUGHES.

Stephen Timoshenko—60th Anniversary Volume. Macmillan, New York, 1938. vii+277 pages. Price $5.00.

As a signal honor in commemorating the 60th anniversary of Professor Timoshenko (December 22, 1938), the Macmillan Company has published a collection of 28 new research articles on the *Mechanics of Solids*. Professor Timoshenko (now professor of Theoretical and Applied Mechanics at Stanford and formerly of Michigan, with continuous connections with Westinghouse) certainly needs no introduction to mathematicians. Detailed arrangements for this publication were made by a committee consisting of Professors J. M. Lessells (M. I. T.), J. P. Den Hartog (Harvard), G. B. Karelitz (Columbia), R. E. Peterson (Westinghouse Research Laboratories), and H. M. Westergaard (Harvard) men who are also numbered among the contributors.

Apart from their significance in honoring Professor Timoshenko, these articles in themselves are meritorious. To the mathematician who is perhaps not primarily interested in their technical content, their most interesting feature is the frequent use of such procedures as ordinary and partial differential equations and single and double Fourier series, and to a lesser extent, harmonic functions and orthogonal functions.

It is a welcome sign in this troubled world when 28 men from many different countries can pay homage to a man originally of still different nationality. This book and the presentation of the Lamme Medal last summer by S. P. E. E. both attest to the esteem held for this outstanding man of engineering, mechanics, and applied mathematics. The Macmillan Company is to be congratulated for its part in making possible the publication of this volume.

North Carolina State College. JOHN W. CELL.